The Mathematics Teacher

NOVEMBER 1961

The Mathematics Teacher

is an official journal of The National Council of Teachers of Mathematics devoted to the interests of mathematics teachers in the Junior High Schools, Senior High Schools, Junior Colleges, and Teacher Education Colleges.

All editorial correspondence, including books for review, should be addressed to the Editor.

All other correspondence should be addressed to

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N.W., Washington 6, D.C.

Printed at Menasha, Wisconsin, U.S.A. Second-class postage has been paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412, P. L. & R., authorized March 1, 1930. Printed in U.S.A.

The Mathematics Teacher

volume LIV, number 7 **November 1961**

THE MATHEMATICS TEACHER is published monthly eight times a year, October through May. The individual subscription price of $5.00 ($1.50 to students) includes membership in the Council. For an additional $3.00 ($1.00 to students) the member may also receive *The Arithmetic Teacher*. Institutional subscription to either journal: $7.00 per year. Single copies: 85 cents each. Remittance should be made payable to *The National Council of Teachers of Mathematics*, 1201 Sixteenth Street, N.W., Washington 6, D.C. Add 25 cents for mailing to Canada, 50 cents for mailing to foreign countries.

Mathematics from the modern viewpoint

TRUMAN BOTTS *and* LEONARD PIKAART. *University of Virginia, Charlottesville, Virginia.*

The axiomatic approach used in plane geometry can also be applied to numbers, and, in fact, to all of mathematics.

IN THIS ARTICLE we are mainly concerned with a point of view and its recent history. This point of view, the axiomatic one, is itself easy to grasp and is already more or less familiar. What is not so widely realized is that all of mathematics can nowadays be approached in this illuminating way, even the mathematics underlying the familiar arithmetic manipulations of elementary school.

THE AXIOMATIC APPROACH

Twentieth-century mathematics has been, and is, growing with unprecedented speed and vigor in a number of directions. It is of course important for well-informed people in general, and teachers of mathematics in particular, to know something of these various new developments. However, if we had to pick out a single development which is most nearly the key to the modern spirit in mathematics, it would be the *widespread application of the axiomatic approach.* The axiomatic approach itself is far from new: it goes back to the Greeks of the fourth century B.C. and earlier. But the effective application of this approach to literally all of mathematics is largely a development of the twentieth century.

Everyone has a rough idea of what is meant by the axiomatic approach to a subject. In this approach we clearly set forth at the outset (1) the basic notions to be dealt with and (2) the properties these notions are to be assumed to possess. The statements asserting these assumed properties are usually called *axioms.* We then proceed, defining new notions as needed in terms of the basic ones and deducing new assertions, called *theorems,* from the axioms and from previously proved theorems by the methods of logical deduction. The basic notions in the axiomatic approach to a subject are sometimes called *undefined* notions to emphasize our intention not to take for granted any properties of these notions other than those specifically assumed in the axioms. Nothing, however, prevents our choice of axioms from being guided or motivated by our intuitive ideas about the subject.

This will all be clearer if we consider an example. Probably for most people the nearest thing to a familiar example of the axiomatic approach is the ordinary plane geometry of high school. There is an additional reason for choosing this particular example. It stems from Euclid's *Elements of Geometry* (300 B.C.), which was the first, and for many centuries virtually the only, sustained exposition of a subject from the axiomatic point of view.

THE EXAMPLE OF GEOMETRY

Mathematicians have found that in the axiomatic approach to plane geometry any of several sets of basic notions can serve equally well as a starting point. For example, one way we can begin is with the basic notions *point, line,* and *between,* where "between" in this technical sense is to be used only in contexts like this: "Point A is between point B and point C."

Of course, when we begin in this way, our choice of terms reflects the fact that we have in mind certain physical objects for which we would like to think of our mathematical points and lines as idealizations. Thus the term "point" suggests a small chalk dot on the blackboard. The term "line" suggests a long, straight chalk mark. And the statement "Point A is between point B and point C" suggests some such picture as this.

Our choice of axioms, like our choice of terms, is motivated by our hope that the geometry we are constructing will prove useful as a kind of idealized *model* for certain features of the world of physical experience. With *point, line,* and *between* as basic or undefined notions, some of the typical axioms follow. Note that each of these axioms is a reasonable abstraction from our physical experience.

Axiom 1. Every line is a set of points.

Axiom 2. For every two points there is exactly one line containing both of them.

Axiom 3. If point A is between point B and point C, then point C is not between point A and point B.

As mentioned earlier, new notions are to be defined as needed in terms of the original undefined ones. An example would be the following definition of the notion of *line segment.*

Definition. If A and B are two points, the *line segment* AB is defined as the set of all points C such that C is between A and B.

The *theorems* of our geometry consist of all the assertions about the undefined notions (and other notions defined in terms of these) that can be deduced logically from the axioms and the theorems already deduced. The following is a simple example of a theorem that can be deduced using only the axioms we happen to have listed above as samples.

Theorem: If A, B, and C are three points and A is between B and C, then C is not contained in the line segment AB.

Proof:

[1] A is between B and C.
(By hypothesis)

[2] C is not between A and B.
(By [1] and Axiom 3)

[3] Therefore, C is not contained in AB.
(By [2] and definition of AB)

Pure and Applied Mathematics

In our first section, we described briefly what is meant by the axiomatic approach to a subject, and in the second section we illustrated this with the familiar example of plane geometry. Let us review the component parts that make up the axiomatic treatment of a subject. These are

1. undefined notions,
2. axioms,
3. definitions,
4. theorems.

We have already indicated and illustrated the role of each of these component parts.

We have also pointed out that the terms used for the undefined notions and the assumptions chosen to serve as axioms may be (and in the above example of geometry certainly are) suggested by our experience with the physical world. At the same time, we should keep clearly in mind that the undefined notions themselves—the things the axioms and definitions and theorems are actually *about*—are not physical objects at all: they are just abstract objects of thought.

The undefined notions, the axioms, the definitions, and the theorems that go to make up the axiomatic treatment of a subject are sometimes said to form a *mathematical system.* Thus plane geometry is one mathematical system. As we shall see shortly, there are many others. According to twentieth-century views, the collection of all mathematical systems is by definition what we call *mathematics* or *pure mathematics.*

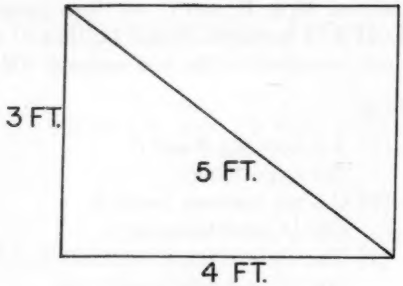

Figure 1

So-called *applied mathematics* is what results when the abstract notions of a mathematical system are replaced by concrete physical notions. When this is done, the axioms and resulting theorems are transformed into assertions about some part of the physical world, to be verified (often just approximately) by the methods of physical observation and measurement. For example, the following paragraph outlines a typical simple application of plane geometry.

A table top is rectangular, and two of its adjacent sides have lengths of 3 ft. and 4 ft., respectively. Therefore, since the area of a rectangle is the product of the lengths of any two adjacent sides, the area of this table top is $3\times4=12$ sq. ft. Furthermore, since in a right triangle the square of the length of the hypotenuse is the sum of the squares of the lengths of the other two sides, and since $3^2+4^2=5^2$, the diagonal distance across the table top is 5 ft. (Fig. 1).

Historical remarks

Today the advantages of the axiomatic approach to a subject for clear, orderly thinking seem obvious. Such an approach lays bare, so to speak, the logical structure of the subject. At the same time, it is in ideal form for application, since all the hypotheses used in arriving at its conclusions are clearly stated at the outset. Nevertheless the mind of man was a long time in seizing on the axiomatic approach and even longer in perceiving its tremendous scope. The Euclidean geometry of the Greeks of the fourth century B.C. was the first great achievement of axiomatics. It is not very far from true to say that there were no further significant advances in axiomatics until the nineteenth century.

During the intervening centuries, the axioms of Euclidean geometry themselves came under more or less intensive study. The principal object was to try to deduce Euclid's axiom on parallel lines from the other axioms. In a modern form this axiom reads as follows:

Parallel axiom. For every point not on a given line there is exactly one line which contains this point and is parallel to this given line.

The effort to prove that this axiom had to follow logically from the other axioms was doomed. In the early nineteenth century it was independently discovered by K. F. Gauss (1777–1855; German), J. Bolyai (1802–1860; Hungarian), and N. I. Lobachevski (1793–1856; Russian) that new geometries quite as consistent as the Euclidean one resulted when Euclid's parallel axiom was replaced by the following axiom (and the other Euclidean axioms were retained unchanged).

Axiom. For every point not on a given line there are at least two lines which contain this point and are parallel to this given line.

Geometries which differ from Euclidean geometry in this way are called *non-Euclidean.* So are the geometries obtained by replacing Euclid's parallel axiom by the following axiom:

Axiom. Every two lines contain in common at least one point (i.e., there are no parallel lines).

This second kind of non-Euclidean geometry was discovered by B. Riemann (1826–1866; German). For additional elementary information concerning non-Euclidean geometries, one may refer to M. Richardson, *Fundamentals of Mathematics* (Macmillan, 1941), Chapter XIV, and to the references given there.

The possibility of such consistent alternatives to Euclidean geometry helped

to widen the horizons of mathematics and to focus attention anew on the axiomatic method. There were several important results. For one thing, a clearer understanding of the axiomatic approach itself was evolved. From early times on, axioms had sometimes been thought of as "self-evident truths," "statements too evident to require proof," and so on. By the end of the nineteenth century, however, the character of axioms as outright assumptions or hypotheses was generally recognized by mathematicians. By this time, too, it had been made completely clear and explicit that the basic notions of geometry (or of any mathematical system) are abstractions, undefined except to the extent that their properties are specified by the axioms.

Another important result of the resurgence of interest in the axiomatic approach was the successful application of this approach in other parts of mathematics. In particular, the various number systems of mathematics were for the first time systematically treated from the axiomatic point of view.

The axiomatic approach to numbers

The simplest numbers are the so-called *counting numbers* or *natural numbers,*

$$1, 2, 3, 4, 5, \cdots.$$

In the year 1889, G. Peano (1858–1932; Italian) gave the first axiomatic treatment of the natural numbers. It is instructive to take a brief look at Peano's axioms. These axioms, only four in number, are very simple. (Historically there were *five* axioms in Peano's original system. We have obtained four by the simple expedient of combining two of Peano's axioms in our single Axiom 2.) We shall try to develop our discussion in a way that will indicate how these axioms might have been arrived at in the first place.

We all have some intuitive grasp of the natural numbers

$$1, 2, 3, 4, 5, \cdots$$

and their properties. Now we propose to set up a system of axioms for the natural numbers *viewed as undefined objects.* Our problem is to abstract from our intuitive ideas concerning natural numbers the basic properties we shall wish the axioms to postulate.

Perhaps the most striking feature of our intuitive conception of the natural numbers is that these numbers occur in a certain *order of succession.* We might symbolize this schematically as

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow \cdots$$

Figure 2

Let us try to express in an axiom this property that each natural number has a *next,* or *succeeding,* natural number.

Axiom 1. To each natural number n there corresponds another natural number called the *successor of n.*

Of course this axiom alone is not sufficient to force our undefined objects, the natural numbers, to succeed one another in precisely the way we have in mind in Figure 2. For example, twelve objects which succeed one another in the "circular" order of succession shown in Figure 3 would be entirely in accord with Axiom 1.

The order of succession illustrated in Figure 3 is even a useful one: it is the order of succession of the twelve hours of the

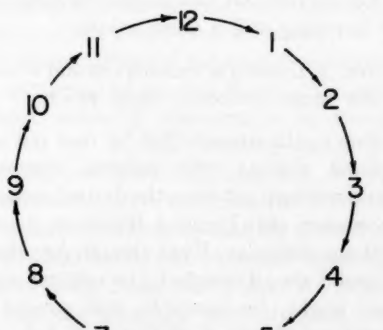

Figure 3

day. But it happens not to be the order of succession that our intuitive ideas prescribe for the natural numbers.

Let us compare Figures 2 and 3 for essential differences. One essential difference is this: each number in the succession in Figure 3 is the successor of some other number; but in the desired succession shown in Figure 2, there is one number, 1, *which is the successor of no number*. Let us phrase this new requirement in a second axiom.

Axiom 2. There is exactly one natural number which is the successor of no natural number. We denote this natural number by 1.

Must our undefined objects now succeed one another in exactly the desired way? No, for an order of succession like the one suggested in Figure 4 would still satisfy both Axiom 1 and Axiom 2.

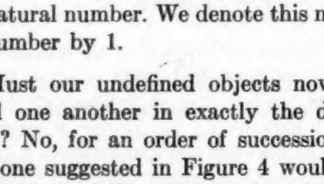

Figure 4

The trouble here is that two of our numbers, 3 and 7, *both have the same successor*, 4. Let us rule out this undesired behavior by imposing still a third axiom.

Axiom 3. If natural numbers m and n have the same successor, then $m=n$.

One might suspect that by now our undefined objects, the natural numbers, could not help but have the desired order of succession, as in Figure 2. However, there is still one difficulty. Even though Axioms 1, 2, and 3 are all satisfied, the natural numbers might, for example, still consist of undefined objects $A, B, C, 1, 2, 3, 4, 5, \cdots$ with an order of succession which is "in two pieces," as shown in Figure 5.

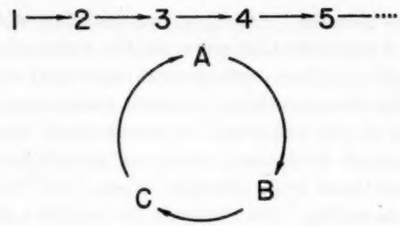

Figure 5

To rule out the unwanted "second piece," we adopt a fourth axiom.

Axiom 4. If a set of natural numbers contains the natural number 1 as a member and also contains the successor of each of its members, then it contains all natural numbers.

We have now stated four axioms for the system of natural numbers, considered as an abstract mathematical system. It is worth observing that in addition to the notion *natural number*, the notion *successor* should itself be viewed as an undefined basic notion in this system. In fact, like "between" in geometry, the term "successor" is to be used technically only in certain contexts—namely, contexts like the following: Natural number m is the successor of natural number n.

Though we shall not do so here, it can be shown that all conceivable "bad" possibilities (such as those suggested in Figures 3, 4, and 5) have now been ruled out. With all four of the above axioms satisfied, it turns out that the system cannot help but have the desired structure, that suggested in Figure 2. In this sense, the natural number system is completely defined, or determined, by these four axioms.

Further axiomatic developments

We have just presented Peano's formulation of the natural number sequence

$$1, 2, 3, 4, 5, \cdots$$

as an abstract mathematical system having two undefined notions—natural number and successor—and four axioms.

There are many important things about the natural numbers that have not appeared in this axiomatic formulation. For example, the operations of addition and multiplication and their properties are not considered. If Peano's axiomatic formulation is a truly adequate one, it should be possible to define addition and multiplication in terms of the undefined notions of natural number and successor. It should also be possible to deduce as theorems the various familiar properties of these operations.

As a matter of recent mathematical history, all this and more has been done. In fact it has all been organized, with rather relentless precision, in a book that has become a modern classic of rigorous mathematical exposition. Its author is the late distinguished German mathematician, Edmund Landau, and its title is *Foundations of Analysis* (or, in the original German, *Grundlagen der Analysis*). Published initially in German in 1930, this book has now been translated into many languages, including English.

Landau does not stop at developing the properties of the natural number system itself. He also shows in detail how, starting with just the natural number system, one can build up, successively, the systems of fractions, rational numbers, real numbers, and complex numbers. In this way, all the common number systems of mathematics are exhibited as depending ultimately on nothing but Peano's simple axiom system together with the processes of logical deduction.

As advanced mathematics students know, it can become a lengthy and rather tedious task to wade through Landau's treatment. However the initial details are easy and fascinating to work through for the first time. One quickly finds that the axiom we have labeled Axiom 4 in the preceding section is a powerful tool. Some readers may recall from high school how to prove formulas like

$$1+2+3+\cdots+n=\tfrac{1}{2}n(n+1)$$

by "mathematical induction." If such a reader is acute, he may recognize Axiom 4 as the underlying principle used in such proofs.

In the last part of this discussion we have been taking a glance at the mathematics of numbers viewed as an axiomatic development comparable with that of geometry. It is a fair question, which mathematicians asked themselves early in this century, just how sweeping can such axiomatic developments be made? Can all of mathematics be subsumed in a single axiomatic structure? Can notions like point and natural number be profitably defined in terms of even simpler and more basic notions?

The answers to these questions turned out to be "Yes." As a result of investigations spanning roughly the first four decades of this century (investigations in which the names Zermelo, Fraenkel, Von Neumann, Gödel, and Bernays were especially prominent), we now know that all of mathematics can be viewed as stemming from a single axiomatic system called *axiomatic set theory*. In this system, the sole undefined notions are *set* and *member*, where "member" is to be used technically only in contexts like:

Set A is a member of set B.

All the other notions needed in mathematics can be defined in terms, ultimately, of just these two; and all the theorems of mathematics can be derived, ultimately, from just the axioms for set theory.

Perhaps the most impressive exposition of mathematics on this basis is the celebrated series of monographs in French written by various authors under the common pen name N. Bourbaki. (For an entertaining historical account of the N. Bourbaki pseudonym, see an article by P. R. Halmos in the *Scientific American* magazine for May, 1957.) Here we cannot go into the details of this sweeping view of mathematics as a single grand axiomatic system. Still it is certainly worth while for anyone with an interest in modern mathe-

matics to know that such a view is possible. It is also instructive to notice the basic role played by the notion of *set* in the sweeping axiomatic formulation of all mathematics. This helps to explain why mathematicians today are so much interested in sets.

Suggested reading

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Letters to the editor

Dear Editor:

Below I have described an experience I had in class. I have noted amusing incidents of this sort in The Mathematics Teacher and thought you might be interested in this one.

In discussing the derivative of $y=\sin^{-1}x$, I wrote $\sin y=x$ on the board assuming the students knew the definition of the first function. One boy, however, raised the question, "How did you get from the first equation to the second?" He said he understood that I was able to do it because of the (-1) power but wanted me to explain the algebra of it. Later while working with another class on inverse functions it occurred to us that we could present the following farcical proof of the operation:

(1) $y=\sin^{-1} x$.	(1) Given.
(2) $y=\dfrac{1}{\sin} x$.	(2) Definition of a negative exponent.
(3) $\sin y=1x$.	(3) If equals are multiplied by equals the results are equal.
(4) $\sin y=x$.	(4) The product of a number and 1 is that number.

One of the observers noted that this proof was valid only when $\sin \neq 0$. In this world, however, there is never an absence of sin so it is non-zero. Another said we could not be certain that "*s*" and "*i*" were also raised to the (-1) power so it might be necessary to use parentheses around sin.

Another proof of the same statement:

(1) $y=\sin^{-1} x$.	(1) Given.
(2) $\sin y=\sin (\sin^{-1} x)$.	(2) If two numbers are equal then $\sin x=\sin y$.
(3) $\sin y=\sin^{1+(-1)} X$.	(3) Law of Exponents.
(4) $\sin y=\sin^0 x$.	(4) Any number added to its inverse results in zero.
(5) $\sin y=1\cdot x$.	(5) Any number raised to the zero power is one.
(6) $\sin y=x$.	(6) See (4) above.

Thomas M. Mikula
Phillips Academy
Andover, Massachusetts

The geometry of space and time[1]

EDWARD TELLER, *University of California, Berkeley, California.*
The mathematical concept of an invariant is here applied to time and space.

I AM HAPPY to be here at the IBM Symposium. All of you know what IBM is doing, but I must confess that while I know what IBM is doing, I've never understood what IBM is doing. My problem with IBM is this. They make machines. And they make machines which can reproduce anything that our brains can do the moment that we put ourselves in the position of saying precisely what it is that our brains really do. Of course there are certain things in this world about which we do not know—for instance, interest. I haven't yet seen clear-cut evidence that a machine can be interested, but maybe that will come, too. I can tell you what learning is, and one can make a machine to learn. One can make a machine act at random, and whenever a machine can do something, it can do it faster than you. But I have the feeling that what a machine is taking away from you is the kind of thing that you may not be too eager to do anyway, like adding long rows of numbers. The machine, by being able to do whatever you tell it to do, actually presents a very particular challenge because it forces you to think about the extraordinary.

Now all of us have heard the statement that while we are doing a good job in this country on science education, we are not doing a good enough job. I am sure that this is true, and I won't belabor the point. I would rather concentrate on one particular facet of our problem of education—our problem of science education. It is an empirical fact that in certain areas of science the young people make the greatest progress. Among the scientists who, in my field, physics, have made perhaps the greatest contributions are Newton, Eintein, and Niels Bohr. All of them did their really essential work before they were 30 years of age. In our system of education, you don't start to function as a scientist until you are 24, or 23, or at the very best, 22. In my opinion, you have wasted half of your most productive time due to our system.

I would like to give the example which —in my mind—is probably the most famous of all, that of Gauss. I do not hesitate to say that Gauss probably was the greatest mathematician who ever lived. I do not know how many of you have heard the charming story of the occasion when Gauss's mathematical talent was discovered. The story goes that it was in the first grade in Göttingen, and the first-graders misbehaved. As a punishment, they had to do something dreadful. They had to add up all the numbers from 1 to 100. And then when they did it, they were instructed to carry up their slates to the teacher's table and put them face down on it. Well, Gauss sat there, six years old, looked up in the air and then wrote down on the slate 5050, carried it up, face down so that the teacher didn't see it—and the teacher was mad! Gauss obviously shirked —he obviously didn't do any work—he just wrote down a number. And then one by one the students trickled up and put down their slates and the teacher looked them through. There was one number that

[1] This lecture was presented at the opening session of the IBM Junior Science Symposium on October 11, 1960. The Symposium was cosponsored by Columbia University and the Science Manpower Project, Teachers College.

was correct, the number supplied by Gauss. You know how he did it: 100 and 1 is 101. 99 and 2 is 101. 98 and 3 is 101. 51 and 50 still 101. So it's 50 times 101, which is 5050.

Well, instead of getting a scolding, Gauss got a teacher in mathematics. By the time he was 18 years old, he was keeping a notebook as a university student, and in that notebook you will find the most important things that Gauss ever did in his life. Had he done nothing more than what is in that notebook, he would still be the greatest mathematician—at 18 years of age. In this notebook you will find the beautiful and simple method of how to deal with imaginary numbers in the complex plane, a problem which had before that time puzzled all of the greatest mathematicians. In fact, in this notebook you will find one discovery which Gauss never published because he thought that it would upset too many people. Thirty years later, a Hungarian and a Russian, Bolyai and Lobachevski, published the discovery of non-Euclidean geometry. Nobody paid any attention to them because it was too incredible. Only after Gauss's death, when his notebook was unearthed, did people say, "Well, Gauss believed it; it must be true!" But he did it when he was 18 years old!

I do not know what is it that you have and I don't. Maybe the ability to concentrate; maybe the love of puzzles; maybe just that your minds don't have these horrible, deep grooves which mine has by now and which make it more difficult to accept new ideas. But I feel that it is extremely important for all of us to try to see that we find out about the really new things in this world at the earliest possible age when our minds are open to new ideas and when we try to spend three days thinking of but one thing. I think that we are making a great mistake in teaching by trying to teach everything, by trying to give complete proofs and derivations, by trying to ask that the students should know the answers right away and precisely. We should be satisfied with stimulating and letting the students find out by themselves a little more. This is a rather general statement. I shall have a short time to talk to you here, and instead of talking about education to any greater extent, I would like to talk about one of my favorite subjects and about some of its applications.

I mean relativity, a subject which has surprised the scientific world, which caused a revolution more than half a century ago, and about which 99.44 per cent of the people still don't have a ghost of an idea. I claim that we are, in a way, as illiterate scientifically as if we did not know that the earth is round. And why? Because instead of holding up a globe as we do to our children when they are 5 years old or 4 years old, telling them this is the earth—practically when they learn the word "earth"—we go ahead with detailed arguments and try to give complete proofs that students who have studied Coriolis forces and all kinds of other complicated things can appreciate, but which on first approach are not appreciated.

Now I know that all of you can easily appreciate what relativity is. But I would like to talk to you about it in my own simple, naïve manner in which I am not proving anything to you, in which I just would like to make a few general statements. I will try to tell you how these general statements hang together, how they explain what is around us in a particularly simple way, and how they lead to simple conclusions which seem to contradict common sense but which nevertheless have proved to be exceedingly fruitful. I am the more anxious to talk to you about these subjects because it seems to me, perhaps wrongly, that a certain style of doing science is not as strong today as it used to be. Why? I do not know. It used to be that a single person—like Newton, Einstein, or Bohr—working with nothing, playing with nothing but his own thoughts, could find really new, really surprising things, things that seem paradoxical and

yet in the end turn out to be true.

Today, we are often involved in great and extremely important enterprises. They are enterprises that, while of great practical importance, have results that fit in nicely with what we already know. They are not as beautifully unexpected as the statements of relativity. And I want to talk to you about that. Furthermore, because in case you have not fully realized how crazy relativity is, you ought to do so before you are completely grown, for by that time it may be too late.

I would like to start from an exceedingly simple mathematical concept with which you probably are all very familiar, the concept of an invariant, a quantity which will not change if you make certain changes in your problem or in your situation. Such a quantity—and I will mention only one—is, in plane geometry, the distance r between two points. In plane geometry if I put down a co-ordinate system and if I have a point $P(x, y)$ whose position is determined by the x-co-ordinate and the y-co-ordinate, then the square of the distance of this point from the origin as you all know very well can be written as $r^2=x^2+y^2$. (See Fig. 1.) Excuse me for the trivial example: r^2 can be obtained by adding x^2 and y^2 according to the wisdom of Pythagoras. Now you can use a new co-ordinate system, the one with primes in the figure. And if you use this new co-ordinate system, then the same point P will have a new x-co-ordinate, call it x', and a new y-co-ordinate, y'. But the squares of the new co-ordinates x' and y'

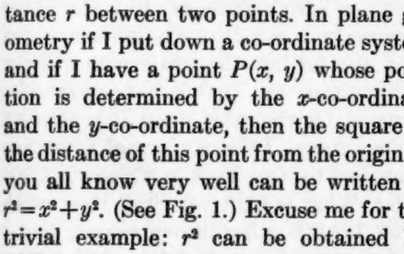

Figure 1

will still add up to the same quantity, r^2,

$$r^2=x^2+y^2=x'^2+y'^2. \tag{1}$$

That is why we say that r^2 is an invariant. It's that simple, so let's not think about something more complicated when I talk about invariance.

Now, then, instead of geometry in space, let's talk about geometry in space and time, which deals with what we call events. An event is obviously characterized by space co-ordinates and by the time at which it happens. Will you please realize one little point? If I take two events, then there is a distance between the two events which I will call r. There is also a time that passes between the two events which I will call t.

Will you first of all realize that the distance r depends on the observer? For instance, yesterday I left San Francisco at 6 o'clock your time. I arrived in New York at 11 o'clock your time. My departure and arrival have taken place as far as you are concerned 3,000 miles apart. As far as I am concerned, the two events have taken place in the same location, namely, in the airplane. If the airplane had moved, which it didn't, quite uniformly on a straight line, which it didn't—and couldn't have because of the curvature of the earth —then I would have been just as justified as you in saying the two events have occurred at the same place, $r=0$, whereas you say $r=3{,}000$ miles. And to show that neither of us could prove his case, what about the observer who stands outside in the planetary system on the sun and says you are both wrong because in those five hours the earth moved so many miles, many more than the 3,000? And what about the person who stands outside the solar system and stands in the middle of our galaxy, with respect to which the sun is moving at more than 100 miles a second? For him, the two events are even farther apart. Who is right? In other words, if I talk about events, the distance between two events, r, is no longer an invariant.

However, it used to be true that t, the

time difference, was an invariant. I left at 6 o'clock. I arrived at 11 o'clock. The time difference is five hours, and it is five hours to everybody. And this was undoubtedly true; it was one of the cornerstones of science, of physics, of common thinking up to the year 1905. And since that time it hasn't been true, and it never will be true again.

Einstein, as you well know, discovered that the time difference is not an invariant, it is not the same for all observers. Now to tear down such an old notion is frequently a very important wrecker's job. It is very important to know what one has to clear away in order to start something new. But the really important thing is to know what you want to put there instead. And what Einstein put down instead is a new invariant, an unsuspected invariant. It is this quantity: take the time difference t between two events multiplied by the light velocity c; therefore, take the distance that light could have traveled between those two events ct. Then take the square of this. Then take the distance r which a certain observer claims between the two events and square that. Take the difference between these two quantities. What you get is an invariant I:

$$I=(ct)^2-r^2. \tag{2}$$

Notice, it used to be true that r changes but t does not. Now r changes and t does also, but the new quantity I does not. Anybody who knows the elements of mathematics must admit that this is a simple enough statement. It is no more complicated a statement than the statement that the earth is round. Yet this simple statement essentially contains Einstein's great discovery.

Those of you who imagine that the theory of relativity is something complicated are completely mistaken. The theory of relativity is something exceedingly simple. The difficulty with it is not that it is complicated, the difficulty is that it violates common sense. That a time span should appear different to you and me, who can believe that? It contradicts everyday experience. We cannot imagine it; and when something contradicts common sense, then the human reaction is to stop thinking it. You forget it. You cannot assimilate it. You say, "If this is true, anything can be true."

But if you will be a little patient we shall treat relativity like the four blind men who attempted to find out about the elephant. They took hold of his trunk and sides and legs, and each of them described the elephant differently. But if you go all around the elephant and find out where the ears are attached, then by and by you become familiar with the animal and you know what the elephant is all about.

So let us begin to see what this expression $I=(ct)^2-r^2$ says. It violates common sense. For one observer, t may be at zero, which means that he will say that the two events are simultaneous. But the other observer will say that t is different from zero, and for him the two events will not be simultaneous. That is how crazy relativity is. On the other hand, it is after all not so crazy. Take my travel from San Francisco to New York in which r is 3,000 miles. What is light velocity? How much is it? 186,000 miles in one second, and I took five hours, or 18,000 seconds. So ct is 186,000 times 18,000—many millions of miles. If I take the square of this quantity, it is enormously bigger than the square of the quantity $(3{,}000)^2$. Therefore, by saying that this whole quantity $(18{,}000 \times 186{,}000)^2-(3{,}000)^2$ must remain unchanged, the quantity $(3{,}000)^2$ practically does not count. So what really has remained unchanged is the time difference which now agrees fortunately with common sense.

So what am I talking about? If I'm talking about these small differences, why am I making all this fuss?

First of all, there is one nice little point about all of this. If the invariant is something containing the square of time and space, then what remains an invariant is much more like what remains invariant

when I consider x- and y-co-ordinates. Therefore, the new theory has a much greater symmetry between time and space.

Time and space now begin to look as two co-ordinates in space looked. This is not quite true because of the minus sign that appears in the relativistic expression.

But there is another and more important point. Please take two events, the snap of my fingers and the fall, a second later, of a meteor on the moon. Let us say these are two events—r distance from here to the moon, time difference, a second. Actually the moon is one light-second away.[2] Therefore, for these two events r is equal to ct. All right? Then r is 186,000 miles, t is one second, and c times t is 186,000 miles. Therefore, for these two events, the invariant is zero.

Now let somebody else look at these same two events, somebody for instance who is going fast from here toward the moon. Let us say that in this one second he has covered half the distance. He has to go very fast to do that, but he might do it. Therefore his r would no longer be the distance from here to the moon but only half that distance. That is what you would expect. But then for him the time difference must also have decreased because only in such way can $(ct)^2-r^2$ still be zero and ct still be equal to r. He will find as I did that his new t' and r' are still connected in that ct' is equal to r'. In other words, if he measures the light velocity, it comes out to be the same, c. He has run after light with a velocity one-half of the velocity of light in the hope that he may begin to catch up with light. But he doesn't. He still sees light going ahead of him with the same old speed. Absurd? It is. But experimentally true. Nobody could run that fast these days! But if we run as fast as we can, namely with the orbital velocity of the earth, one can see what difference that makes to light, and it makes no difference to light with exceedingly great accuracy—it was this one simple hard fact which forced Einstein to make this apparently crazy statement.

[2] This is not exactly true, but let us pretend that the statement is exact.

Now let's look at a few other consequences, other things that have grown out of these simple relations. I am trying to tell you as much about simple things in as short a time as I can. What I am going to tell, you may not understand, partly because you feel that there is much more to be said, and that's true; partly because you will obviously notice that I am not giving you proofs. You will have to look for the proofs. I will be satisfied if you understand clearly what I am talking about. You have seen that distances between events and times between events are connected in a surprising manner. Now, the beautiful thing in relativity is that just as time and distance are connected, so quite a few other things are also connected. For instance, energy and momentum are connected.

Energy and momentum both have a simple property; namely, they are both conserved. Perhaps there is a connection between them. Einstein says there is: pick the following quantity pertaining to a moving body; take its energy E; square it, and then subtract from it the square of the momentum p of the body multiplied by the square of light velocity c: $E^2-(pc)^2$. Again this difference is an invariant. Now this is a particularly amusing statement. This simple formula (it is not even an equation but just a statement that we have an invariant) is connected with three conservation laws. To make this quantity $E^2-(pc)^2$ an invariant, one has to assume that the energy does not become zero if the body is at rest; rather it becomes equal to the mass of the body times the square of the velocity of light, mc^2. The quantity $E^2-(pc)^2$ will be equal to $(mc^2)^2$. If the body is at rest ($p=0$), you see that the formula

$$E^2-(pc)^2=(mc^2)^2 \tag{3}$$

reduces to the statement: energy is equal to mc^2.

In that approximation then you consider only the rest energy, and the conservation of energy reduces to the conservation of mass.

If you then consider that the energy really is a little different, you will find that the energy has been increased not exactly, but almost exactly, by what you usually call the kinetic energy of a body. Try it out; it will work. (See Appendix.) The conservation of this little increment in energy is connected with our classical law of conservation of energy.

Then finally, state that E and p are connected like space and time; to the person who travels with the body, p is zero and E has a certain minimum value. To the person who is at rest and the body is flying at him, E is greater and p is greater. One of these quantities could not have the property of being conserved without the other being conserved also. As the mass is conserved, which is a big part of the energy, then the conventional part of the energy, the excess over mc^2, is also conserved. The momentum is also conserved, and all three are connected with the fact that $E^2-(pc)^2$ is an invariant.

One more thing. Take the simplest case where $(pc)^2$ is zero and then let us perform the operation of extracting the square root.

$$E=\pm mc^2. \tag{4}$$

This suggests that bodies might conceivably exist whose mass is negative. Now this is a paradoxical way of putting things. But what does a negative mass mean? A negative mass means that if I increase the momentum of a body by imparting to it a momentum, by acting upon it with a force in a certain direction, the body will not accelerate in this direction but in the opposite direction, because the mass is nothing else but a ratio between momentum and velocity. Bodies which systematically get accelerated in the direction opposite to that you expect can be said to have a negative mass. If you talk about a charged body, you may say that instead of having a negative mass, the body has the normal mass but the opposite charge.

This is the route along which the so-called antiparticles have been introduced. And we now know that to all kinds of particles out of which matter is composed the opposite particles do exist. We have found the antielectron which is a positron. We have found the antiproton which is a negatively charged proton, and we have also found the antineutron which is still neutral but at least has the opposite magnetic moment, so that under inhomogeneous magnetic forces it will move in the opposite direction. There is no doubt about it that out of these three you could build up antimatter. And we furthermore note that contact of matter with antimatter will result in annihilation, will result in transformation of all of the mc^2, this tremendous amount of energy, into some form of radiation.

We are living in the Space Age, and it would not be quite fitting to conclude a talk of this kind without talking about space travel. There is no doubt that by the time you have grown up thoroughly—meaning by that that you are as old and stupid as I am—by that time the planetary system will be explored. I'm not talking about trivial things like that. I am going to talk about trying to get to the nearest star which is four light-years away. Well, if you take the best fuel we have today, fusion fuel, and if half of your rocket is fuel and the other half of it pay load, you will be able to accelerate not quite to 1/20 of light velocity. You will take 80 years to go to Proxima Centauri. You may lose patience!

Now what can we do? We can have a multiple-stage rocket. We can accelerate the first stage to 1/20 of light velocity and then take a smaller mass and accelerate that to 1/20 of light velocity with respect to the first and go on in this way, stage by stage. The only trouble is that each stage will be smaller than the previous one by a considerable factor, and you are working

on the wrong end of an exponential function. Furthermore, you will never, but never, exceed light velocity.

There is one hope. If we could make antimatter, this would give enough energy to accelerate things to almost light velocity. And we could make antimatter; we already have made the building blocks so we only have to put them together. The real trouble with it is this: where is a container? If you try to contain antimatter in a container, it will disintegrate, right then and there. Well, you could confine antimatter by magnetic fields. Magnetic fields have the beautiful property of behaving the same way with respect to matter and with respect to antimatter. That is obvious and proven and clear. But how long can a magnetic field contain antimatter? Well, we are beginning to get a little experience about that because we are trying these days to use heavy hydrogen as a fuel. To do that, the heavy hydrogen must be at a very high temperature, and this high temperature would be lost in collision with the walls of the vessel. And so we try to confine heavy hydrogen with the help of magnetic fields. If we are going to be successful in that, and we will be, then perhaps we might be on the way—maybe in a hundred years, probably in 500 years—to make enough antimatter and contain it so that we can begin to travel really great distances and go star hunting.

Now having gone this far, I don't suppose that you will be satisfied because you know of course what kind of a world we are living in. We are living in a conglomeration of some hundred billion stars called the Milky Way system. Actually we are leading a somewhat suburban existence in one of the arms of the spiral, 30,000 light-years from the metropolitan center. All around our system there are other Milky Way systems, the closest one about two million light-years away, the Andromeda Nebula. Have we any hope of getting to the center of our galaxy? Do we have any hope of ever visiting Andromeda? I have told you that nobody can go faster than light. Therefore, whoever wants to get to the center of our galaxy must spend 30,000 years, and your family physician will tell you that you don't have that much time! And whoever wants to go to Andromeda needs to spend almost two million years, and that's even more ambitious.

I must admit that the purely engineering difficulties are considerable. However, I claim that the biological difficulties are not necessarily insuperable, and I will tell you that this happens to follow from the invariance of this beautiful formula, $(ct)^2-r^2$. Pretend that Andromeda is two million light-years away, and I will tell you I will travel to it by going just a shade, a slight shade, slower than light. That is the most I can do. Well, if I travel a little slower than light, then I will take just a very short time more than two million light-years to get to Andromeda. That is at least what you stay-at-homes will think. But what will I think? Let me tell you first what you will think in real detail. You will say that the time of my arrival is later than my time of departure by two million light-years plus a shade. Therefore, $(ct)^2$ for my departure and arrival is two million light-years plus a shade. You who stayed at home will then state that $(ct)^2-r^2$ is a small quantity.

Now what will I say? I will say that my departure and my arrival have occurred in the same place, namely, behind the controls of the rocket. So for me r is zero and since $(ct)^2-r^2$ is an invariant, $(ct)^2$ for me should be very small, and I can get to Andromeda in my lifetime.

Of course if I travel to Andromeda, I may get into all kinds of trouble. First of all, there is a real danger that Andromeda might not be a galaxy at all. It may be an antigalaxy. You see, we are pretty sure that our own galactic system does not contain much antimatter. If it did, we would find antimatter in cosmic rays that rain down from all over the place, but only from our own galaxy. A customs barrier is erected around our galaxy in the form of a

magnetic field which will not let out any of the cosmic rays or let in foreign cosmic rays. So we don't know whether outside our own little system barely 100,000 light-years across there is antimatter or matter. In our system we have reason to believe that there is not much antimatter. But Andromeda may be an antigalaxy, and over there on the first contact I may be transformed into radiation.

Then there is a further danger. I may become homesick, and I may want to turn around and come home. Having spent perhaps ten years going and ten years coming, when I come home I might expect a ticker-tape reception. It's not what I'm going to get. You will be dead, all my friends will be dead. On the earth, four million years will have passed. Nobody will speak my language; the human race no doubt will have evolved into a race greatly superior to ourselves. I am afraid that I shall be put in a zoo!

Now, isn't this unjust? Isn't it unjust from the point of view of relativity? I went away, and I came back. But from my point of view I could as a good relativist describe the situation differently. I stayed in the same place all the time, in my rocket ship. It was the earth and the solar system and the galaxy and also Andromeda that went away and then came back! The fact that the nontravelers are in a majority and that I am the single exception who did the traveling should not make, from the point of view of a relativist, any difference. Why is it then that physically I should have stayed young? And you here had the marvelous possibility of evolution in the meantime! Why? Well, there is a physical difference between you and me. And this physical difference is a tricky question.

Einstein puzzled about this difference for thirteen years and then came out with the theory of general relativity which is a great improvement on special relativity. The difference is this: you who have stayed at home on earth did not experience any extraordinary accelerations. Now acceleration is a real thing which influences all measurements, all physical processes. I, on the other hand, when I went to Andromeda, stopped and turned around, had a whale of a lot of an acceleration, in fact one that probably in all physiological justice should have killed me anyway. But let's disregard that because I am talking about physics and not about physiology. It is true that while I am traveling away, if I'm trying to observe what happens on the earth and try to apply all the necessary corrections, I will not admit that my clock is moving slowly. It is you who stayed at home on earth who stood practically still, and in the ten years in which I traveled, only a few seconds had passed here on earth.

But then I stop. And when I stop, then I see a great difference. And let me tell you what that difference must be. You probably have heard about the strange phenomenon, the Lorentz-Fitzgerald contraction, and I will explain it to you.

For me who traveled to Andromeda, ten years have passed. I must see that the earth, the galaxy, and Andromeda all moved by in ten years only—but still with light velocity—because my speed relative to you must be the same as your speed relative to me. Therefore, I am forced to say that the distance between Andromeda and the earth is only ten light-years. For me, the traveler, everything seems to be flattened, contracted. If you don't happen to be able to remember this strange thing, you might remember this limerick, which describes this situation.

> There once was a sprinter in action
> Who lost his best race by a fraction.
> When he went through the tape,
> He altered his shape
> By the Lorentz-Fitzgerald contraction.

Which will remind you that the Lorentz-Fitzgerald contraction unfortunately occurs precisely in the direction in which we are trying to move.

The moment I stop, I notice that the distance between me in Andromeda and you here on earth is no longer ten light-years as it appeared to me while I was

moving. In the procedure of stopping, the Lorentz contraction has undone itself, and while I still see you in your original unchanging state, I evaluate it as being two million light-years away. I appreciate that the way I see you behind me is no longer the present but history, ancient history of two million years' standing.

I will not continue this analysis, but it is clear that violent changes occur while my speed is changing, while I'm accelerating and decelerating. The fact is that while I am turning around I must say time on earth must have passed at a breakneck speed. Now when I turn around what I feel is an acceleration, an acceleration pressing me back toward the earth, an acceleration giving me the same feeling as though I were pressed against the bottom of the rocket by exceedingly strong forces. Let me draw the situation.

Here is the earth, not on scale, and here is Andromeda, not on scale either (Fig. 2). I am in the rocket just beginning to take off, accelerating the rocket toward the earth.

I will feel the rocket pressing against my feet as if there were a giant, a heavy attracting body underneath me in a direction away from the earth. Under these conditions time seems to pass on the earth much faster than time passes here for me in the rocket. Einstein noticed that from this concept of the difference of time passage due to effects like that of acceleration, something might follow. He added a very interesting observation. When you stand or sit in a room, you do not really know whether your feet press against the floor because the earth is attracting you or because all of this is a part of a rocket accelerating upward. This is the principle of equivalence. And from this there followed a conclusion that time must pass differently in different parts of a space in which forces act, whether these forces are due to gravitation or whether they are due to acceleration.

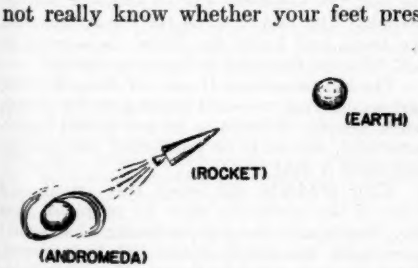

Figure 2

This apparently complicated, far-fetched conclusion has been verified quite recently in the laboratory, where it was noticed that very delicate comparisons of time can be made even between the top of the building and its bottom. Time, if you measure it very accurately, passes at these two positions a little differently. One of you may be talking about this question in connection with the Moessbauer effect.

Now I would like to say just one word in conclusion. From general relativity which links together not only time and space but brings into the system acceleration and gravitation, we have gained a picture of the universe whereby we can describe very distant events, very high velocities. When we look out on the universe, we actually see that the farther we go away, the faster objects travel away from us. From the study of these distant objects which we see now as they have been a long time ago, we may gain some insight into the creation of the universe, if indeed the universe ever was created. I don't believe that it is likely that anyone of us can guess what the history of the universe has been. Usually when you attack problems of this kind, the real solution is simpler and at the same time stranger than anybody can imagine. But I can say this, in Einstein's relativity we have the tool which combined with the most modern methods of astronomy might even in your lifetime tell you whether the universe has ever been created and if so, how.

Appendix

We have the expression

$$E^2-(pc)^2=(mc^2)^2.$$

This expression may be rewritten as

$$E^2=(mc^2)^2+(pc)^2$$

or, taking the square root, as

$$E=\sqrt{(mc^2)^2+(pc)^2}$$
$$=mc^2(1+p^2/m^2c^2)^{1/2}.$$

The term $(1+p^2/m^2c^2)^{1/2}$ may be expanded in a McLaurin's series.

Thus we have

$$E=mc^2(1+p^2/2m^2c^2+\cdots),$$

and this expression is valid for $p^2/m^2c^2<1$. It may be rewritten as follows:

$$E=mc^2+p^2/2m+\cdots.$$

Have you read?

LAKE, KENNETH E. "How To Win an Argument," *School Science and Mathematics*, March 1961, pp. 159–162.

Here is an article that everyone should read. This would end, once and for all, all arguments among people who don't want to find out they are wrong.

The first challenge in an argument is, "Define your terms." With this challenge, you place your opponent deep in the sea of uncertainty.

Is it possible to define words with words? The author has a nice little mathematical proof which says "No," unless our vocabulary is infinite. Mathematics, as you can see, does deal with things nonquantitative. But, we must not lose sight of the assumption we started with; that is, "terms defined with words."

The mathematician wouldn't get caught in this predicament because he will also have some operational definitions.

Better read this discussion and be prepared. —PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

SAWYER, W. W. "The Reconstruction of Mathematical Education," *Journal of Engineering Education*, November, 1960, pp. 98–113.

The question of balance between modern mathematics and classical mathematics goes on and on. Here is a good article directed to the problem. Dr. Sawyer in his concise style lays the cards on the table for you to judge, while at the same time giving you his opinions. For example, the nature of mathematical employment is changing, the classical concepts are threaded through the modern concepts, and the watertight compartments have started to break down. This very activity has led to some confusion, and programs for development have led to frustration. Mathematicians are beginning to realize that good mathematics will not necessarily be good pedagogy. The author is concerned that "modern mathematics" may distract our attention from the real nature of our problems and that we may look to modern mathematics as a glamorous cure for all our problems. His concern for physics, independent study, the place of judgment, intuition, and the ability for seeing the problem is worthy of thought.

All mathematics teachers should read this article.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

SHEPARD, JOSEPH K. "Legislating Mathematics," *Indianapolis Star Magazine*, April 30, 1961, pp. 10, 12, 14.

Do your students ever ask, "How many decimal places shall I use in π?" If so, you will be happy to note that the Indiana Legislature tried to do something about this way back in 1897. House Bill 246 was introduced by Representative Taylor I. Record of New Harmony. He admitted he couldn't understand it, but he trusted his friend, a medical doctor, who wrote the bill. Dr. Goodwin was an amateur mathematician who had squared the circle, trisected the angle, and found the "true" ratio π to be 3.2. All these formulas he had copyrighted.

The bill passed the House, 67 Ayes, 0 Noes, and was having a second reading in the senate when Senator Hubbell, a former school superintendent, moved it be postponed indefinitely, and there it died.

Your students will enjoy this article and some of the comments made by people of that day. Pupils may also try predicting what would have been the result if the bill had passed. What do you think?—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

The psychological appeal of deductive proof[1]

GERTRUDE HENDRIX, *UICSM Mathematics Project, NETRC Mathematics Study, University of Illinois, Urbana, Illinois.*

It is the power of deductive reasoning to yield correct predictions which accounts for the primitive appeal of formal proof.

> If increasing an individual's certainty *cannot* be a motive for deductive proof until after he has acquired considerable skill in the process, then what is the primitive motive for this activity? Can it be the discovery that careful use of precise linguistic formation yields prophetic power, a power of prediction less fallible than the unverbalized deductions usually classified as "hunches"? If so, how can more students be helped to acquire and appreciate this power of language without destroying a taste for it in the process?[2]

THE ABOVE QUOTATION appears in a section "Critical Unanswered Questions" in a monograph recently published by the U.S. Office of Education. The problem area under discussion is that of nonverbal awareness in the learning of mathematics. Since that monograph was written, hypotheses providing clear answers to the questions above have emerged. The hypotheses have been tested with a class of fifteen seventh-grade children in the University of Illinois High School, and a vivid record of classroom results has been recorded on film with sound. Readers who insist that every good story in education be told to them point first, may now read the last paragraph of this article if they so wish, but they are hereby admonished to avoid the temptation.

Most mathematicians attribute the fascination and delight with which they regard formal proof to the confidence and security which they feel when proof of a new conjecture has been achieved. Traditionally, this view has been communicated to those who teach secondary-school mathematics. And so, all over the world, high school boys and girls have been told, "You don't really *know* a theorem until you have proved it." After each proof of a theorem in the early part of the tenth-grade demonstrative geometry course, many teachers have exclaimed, with a kind of pep-session air, "And now, we *really* know it." At this stage of the game, the theorems almost always corresponded with physical space facts easily verified by experiment. Somehow the things called 'proofs', whether found in the book or worked out with the class at the board, have seemed to many students a worthless supplement to previous knowledge. All through the first few months of the course the teacher has been confronted over and over with the cry, "But *why* do we have to prove it when we already know it?"

Near the beginning of the twentieth century in Germany, Felix Klein, a mathematician who was keenly aware of the

[1] The writer is indebted to Max Beberman for the breakthrough to the solution of this problem. Without the interest and initiative and ingenuity with which Mr. Beberman put the "foretelling" hypothesis to a test in a class he taught at University of Illinois High School throughout the year 1959–60, the simple, all-inclusive explanatory hypothesis at the conclusion of this paper would never have emerged. Furthermore, Mr. Beberman's criticisms and alterations of an earlier version of the article have contributed greatly to its interpretability.

[2] Gertrude Hendrix, "Nonverbal Awareness in the Learning of Mathematics," *Research Problems in Mathematics Education*, Cooperative-Research Monograph No. 3, U.S. Government Printing Office, Washington, D.C., 1960, p. 59.

relation between a high school student's experience in mathematics and his capacity for later productive work in mathematics, proposed a deceptively plausible solution to the pedagogical problem of formal proof. He suggested that teachers encourage students to rely upon experimental drawings and inductive intuition until the class reached a point in the development of the subject where deductive proof was actually necessary to settle doubt. Until that time, he advised the teacher to be satisfied with "Behold!" In the United States today, influential groups in mathematics education are urging a revival of this view.

But in the early 1930's, many teachers, guided by Breslich and others, had already given this notion an earnest trial. They found that a proof fails to remove uncertainty for a person who does not already know what a proof does and who has not already had considerable experience in making proofs. To such a student, it does not even seem reasonable to call the argument 'a proof'. Students may have rebelled at "proving" things they already knew, but that situation was not as damaging as the resulting frustration when a class unfamiliar with proof suddenly had to have it to keep afloat in the course. This situation impelled at least one young teacher to burst forth in her first published article in exasperated defeat.[3]

A fairly effective compromise was achieved by the recognition of both experiment and deductive reasoning as ways of knowing.[4] By making it a *game* to know as much as possible of plane geometry by both experiment and reasoning, one could justify proving things the students already knew. Thus enough skill in deduction was developed to provide a tool for future cases in which experiment was impractical or impossible.

At that time, some secondary-school mathematics teachers who were working thoughtfully on these things suspected that the mathematician's delight with deductive proof might be traced to his taste for system rather than to any increase in certainty concerning the things proved. So facile an explanation may have seemed plausible to teachers with limited experience in mathematical research. But producing mathematicians have known all along that this explanation of their confidence and delight in their work was inadequate. There are too many cases of mathematical sentences which mathematicians have tried to prove for decades or even centuries that finally turned out to be false. So we are forced to concede to the mathematician that the grandeur with which he regards deductive proof *does* have something to do with its power to establish the reliability of conjecture—or in the case of an uninterpreted deductive theory, its power to establish consistency and dependence.

Traditional plane geometry textbooks used optical illusions to discredit observation as a way of knowing. This might have been all right had it been balanced by examples of factually false *deductive* conclusions and an analysis of *their* sources: false or inconsistent premises, incorrect applications of rules of inference, and so on. Emphasizing either experiment or deduction to the belittlement of the other weakens the worker. The reliable scholar uses each of these processes as a check upon the other. Often in mathematical research deductive proof is a confirmation of something already grasped by intuition, but not always. Some research mathematicians tell us that when they do arrive at a new theorem by derivation, the first impulse is to try an example, "to see if it really works." Those responsible for mathematics education have never been on tenable ground when holding forth increased certainty and intellectual security as goals to be achieved through deductive proof alone.

[3] G. Hendrix, "A Protest Against Informal Reasoning as an Approach to Demonstrative Geometry," The Mathematics Teacher, XXIX (April, 1935), 178–80.

[4] Barber-Hendrix, *Plane Geometry and Its Reasoning* (New York: Harcourt, Brace, 1937), pp. 1–3.

Nevertheless, there *had* to be some primitive appeal, some motive for inventing and pursuing deductive argument, else this exquisite art of organizing and testing knowledge would never have come into bloom. If it is *not* a quest for increased confidence in something already suspected to be true which causes one to seek and use deductive proof in the first place, why should any human being ever have become aware of deductive proof? What has made man work (or play?) with the process long enough to find out that it *is* an instrument for increasing one's certainty? For the beginning one would probably have to go back to the first tribe who inserted into their language (1) a conditional connective (e.g., the English 'if . . . then'), and (2) words to express future time. A hypothetical explanation was tested and verified with the class whose work was filmed at the University of Illinois High School for the National Educational Television and Radio Center Mathematics Study in 1959–60. The hypothesis is suggested in the quotation at the beginning of this article: *It is the power of deductive reasoning to yield correct predictions which accounts for the primitive appeal of formal proof.*

The development of deductive power revealed by the children in the film class does not seem reasonable—or even plausible—to a viewer of the film record unless the viewer is aware of *the spontaneous nonverbal processes* which correspond with induction and deduction. It is thinking of generalization as fundamentally a nonverbal process which opens the door to the kind of teaching which produces such learning.[5] (The fact that most of the discoveries thus promoted are *re*discoveries does not detract from the dynamic character of the learning which takes place.) When a teacher sets the stage for discovery of a generalization, he must plan some kind of opportunity for the learners to reveal when they have become aware of the desired generalization. The learning exercises must be set up so that the student can begin to apply the new generalization the moment that it dawns upon him. Spontaneous application emerges as evidence that the learner is making deductive use of a newly discovered, unstated generalization. For example, if the discovery of rules for adding real numbers is the objective, the learner shows by the speed with which he begins to write answers in a long list of exercises that he has discovered a short cut. Although the generalization which he is really using in his short cut would be very difficult for him to formulate on the spot, each application of the newly discovered short cut really corresponds to *a deductive inference based on the unstated rule.* A teacher new to this kind of instruction is usually so involved in the difficulties of composing precise questions and wording tactful and strategic disposal of answers that he does not realize the structure of what is going on in the student's mind. What goes on in a learner's mind when he solves a problem by "insight" is not a single process, but rather a discovery of a generalization—that is, an unverbalized *in*ductive inference—followed immediately by *de*ductive application of the generalization.

As soon as a teacher acquires this point of view and the discerning power of observation which it yields, he can begin to exploit the fundamental human desire for ability to foretell. When a teacher knows that the students are aware of a prerequisite principle, he knows that a requested prediction need not be a guess. The principles of arithmetic for numbers of arithmetic, and the principles of arithmetic for real numbers, can be named without the principles themselves being stated. A student can reveal by his ability to construct instances—"Give us another example like these on the board"—that he is aware of a principle, and the principle can be given its name. Students can then be asked to support their prophetic

[5] See G. Hendrix, "Learning by Discovery," THE MATHEMATICS TEACHER, LIV (May, 1961), 290–99.

hunches by naming the principle which supports their predictions. ("I know that '$(9\times 8)\times(7\times 12\frac{1}{2})=(9\times 7)\times(8\times 12\frac{1}{2})$' is true because it follows from the associative and commutative principles for multiplication.")

From this point on, the road is wide open and clear. Soon the same students can acquire linguistic equipment for precise formulation of both premisses and conclusions. Written representations of their thinking then become possible. Very soon the students become aware of the logical rules of inference whose intuitive counterparts they have been using. Written records of inductive and deductive "hunches" make it possible to examine the conjectures with detachment and impartial judgment. Patterns for common mistakes in judgment and reasoning soon emerge. No eulogistic talk about power of proof is necessary to convince these students that premisses (sentences which state the awarenesses from which their hunches spring) can be manipulated mechanically according to logical rules of inference. Neither do the students need to be told that results so obtained are not vulnerable to the disconcerting mistakes which have emerged as false hunches in the past.

Episodes showing all stages in this development have been recorded on films of unrehearsed classroom work. Some of the episodes have been incorporated into the films of the UICSM Teacher Training Series. All of the others have been preserved. The episodes are taken from a full year's work.

* * *

A yearning for foresight is at the very foundation of security for all organic life. Living creatures intuitively and constantly reach out for generalizations which let them know what to expect next. This yearning reveals itself in a somewhat perverted form in one species: witness the preoccupation with fortunetellers and soothsayers threading its way through human history and development! All that a teacher needs to do to establish a motive for acquiring skill in proof is to exploit this primitive goal—that is, to let students find out over and over again that using principles would have enabled them to predict results already arrived at by laborious experimenting and computing. For example, they find that they could have foretold that $36\times 24=30\times 30-6\times 6$, by a logical transformation of '$(30+6)\times(30-6)$'. This foretelling is based on a judicious and systematic application of principles of arithmetic.[6] The students verify several examples by computing. But the test pattern in the footnote below

[6]
$$\begin{aligned}
36\times 24 &= (30+6)\times(30-6) \\
&\qquad \textbf{Principle for Subtraction} \\
&= (30+6)\times(30+-6) \\
&\qquad \textbf{Left Distributive Principle for Multiplication over Addition} \\
&= [(30+6)\times 30]+[(30+6)\times -6] \\
&\qquad \textbf{Distributive Principle for Multiplication over Addition} \\
&= [(30\times 30)+(6\times 30)]+[(30\times -6)+(6\times -6)] \\
&\qquad \textbf{Associative Principle for Addition} \\
&= \{[(30\times 30)+(6\times 30)]+(30\times -6)\}+(6\times -6) \\
&\qquad \textbf{Associative Principle for Addition} \\
&= \{(30\times 30)+[(6\times 30)+(30\times -6)]\}+(6\times -6) \\
&\qquad \textbf{Commutative Principle for Multiplication} \\
&= \{(30\times 30)+[(30\times 6)+(30\times -6)]\}+(-6\times 6) \\
&\qquad \textbf{Left Distributive Principle for Multiplication over Addition} \\
&= \{(30\times 30)+[30\times(6+-6)]\}+(-6\times 6) \\
&\qquad \textbf{Principle of Opposites} \\
&= [(30\times 30)+(30\times 0)]+(-6\times 6) \\
&\qquad \textbf{Principle for Multiplying by 0} \\
&= [(30\times 30)+0]+(-6\times 6) \\
&\qquad \textbf{Principle for Adding 0} \\
&= (30\times 30)+(-6\times 6) \\
&\qquad \textbf{For each } a,\ (-a)\times a=-(a\times a) \\
&= (30\times 30)+-(6\times 6) \\
&\qquad \textbf{Principle for Subtraction} \\
&= (30\times 30)-(6\times 6)
\end{aligned}$$

not only confirms their experiment; it also predicts the outcome for all other cases. Since the same reasons apply to simplifying all expressions of the form:

$$(\square+\bigcirc)\times(\square-\bigcirc)$$

we have derived a short cut to finding all such products. For example, one's hunch that $102\times 98=10{,}000-4$, is now shown to be predictable.

After an adequate repertoire of predictions of things already known has been built up, the teacher may begin to call for predictions before carrying out the verifying (or refuting) experimental solution to a *new* problem.

This development of deduction can, of course, be achieved through the study of geometry. In fact, the appeal-to-prediction hypothesis was partially tested at University High School with a seventh-grade unit on intuitive geometry in the spring of 1958. But the quest-for-reliable-prediction approach, later written into UICSM Units 1 and 2, provides a very teachable access to deductive processes.[7] The principles of arithmetic for real numbers are comparatively few in number; as soon as numerical variables and universal quantifiers are incorporated into the linguistic equipment of the student, these principles are easy to state; and their pattern sentences are easily used for generating instances.

Young students who acquire their introduction to formal proof through a structural treatment of elementary algebra can experience rigor to an extent which has never been attainable in high school geometry. Even the simplest theorems in synthetic geometry have tremendously complicated proofs if one insists upon rigor. So-called 'proofs' for a beginner in geometry must be sprinkled liberally with huge intuitive leaps; the thread of thought becomes lost in a maze of detail if rigor is demanded in geometry.

In contrast, the field properties of the set of real numbers not only can be grasped intuitively; but they also can then be stated (as postulates) with simplicity, clarity, and precision. Their adaptability to algebraic symbolism is what makes such formulation possible. The proofs of theorems in elementary algebra are comparatively brief, and the separation of mathematical premisses from logical rules of inference is distinct.

In such a course, the rigor with which children can become familiar provides an experience which has been denied in the past to all except the few who have had occasion to study formal logic. By the time a mathematics student reaches the university level, there is no time for concentration upon the *nature* of rigor. By that time, instructor, textbook author, and student all take the intuitive leaps for granted. This is all well and good if one *knows* that he is leaping when he does it. The creative mind *must* allow itself to take great leaps if it is to accomplish great breakthroughs; but such a mind is unnecessarily vulnerable to unidentified false landings if the experience of rigor has never been encountered.

* * *

When the University High School film class reached the stage in the course at which they were expected to prove generalizations, they were ready and eager for the necessary formal procedure. This happened early in December, 1959. The sessions being filmed were programmed by Gertrude Hendrix and taught by Max Beberman. The children outdid anything that had been expected of them. Before the eyes of three cameras there came evidence of aptitude, taste, and power for proof, the like of which none of us had seen in beginners before.[8] The aspect of all this which amazes those who look at the evidence is the obvious delight with

[7] University of Illinois Committee on School Mathematics, *High School Mathematics, Units* 1, 2, 3, 4 (Urbana, Ill.: University of Illinois Press, 1959).

[8] For an account of how the films were obtained, see Byrl Sims, "Multicamming Mathematics," *Journal of the University Film Producers Association*, XII (Spring, 1960), 6–8.

which very young students can come to appreciate and use the power of formal proof—"this game played with the thrice-attenuated shades of things."[9]

The steady growth in deductive power for the children in the experimental class can be traced through an entire year in filmed episodes. But it was viewing and editing the early December films that revealed the real basis for faith in deductive proof: It is through regarding derivation as prediction, and thus through finding again and again that things already known by observation and experiment can be proved deductively—this is the way that man acquires his confidence in deductive proof. In the beginning we all need to prove what we already know. Accumulating such experiences provides an *inductive* basis for faith in *deduction*. It is thus that budding young mathematicians come to value deductive proof in the first place. Quite likely it has always been so.

[9] From "Paradox," a poem by Clarence R. Wylie, Jr., *Scientific Monthly*, LXVII (July, 1948), 63.

Letters to the editor

Dear Editor:

I read with interest in the March, 1961, issue of THE MATHEMATICS TEACHER the letter submitted by Gladys B. Rheins concerning the factoring of quadratic polynomials, and it prompts me to send you a few comments of my own.

Too many times students are prone to look for *an* answer to a problem and then look upon their answer as *the* answer. In many instances such is not the case, and this attitude should be discouraged by teachers because there may be one or several answers.

For example, consider the quadratic polynomial $6x^2+32x+32$.

	The factors of *abcd*, 192, are
1. $6x^2+32x+32$	1×192
$6x^2+(24x+8x)+32$	2×96
$6x(x+4)+8(x+4)$	3×64
$(6x+8)(x+4)$ are the factors	4×48
of the polynomial (not prime)	6×32
	$\boxed{8\times 24}$
	12×16

2. $6x^2+(8x+24x)+32$
   $2x(3x+4)+8(3x+4)$
   $(2x+8)(3x+4)$ are the factors of the polynomial (not prime)
3. Using the same quadratic polynomial, but removing the factor 2, it now becomes $2[3x^2+16x+16]$

	The factors of *abcd*, 48, are
$2[3x^2+(12x+4x)+16]$	1×48
$2[3x(x+4)+4(x+4)]$	2×24
$2[(3x+4)(x+4)]$ or $2\ (3x+4)$	3×16
$(x+4)$ are also factors of the polynomial (prime)	$\boxed{4\times 12}$
	6×8

Another example is a word problem of a type common in junior high school. If the difference between two numbers is multiplied by 2, it gives a product of 456. One number is 704; what is the other number? $2x=456$.

Here it is evident the difference is 228. Then 704 can be either the minuend or the subtrahend. If 704 is the subtrahend, then $704+228$ is 932, the other number, and $932-704=228$; $228\times 2=456$. If 704 is the minuend, then $704-228=476$, the other number. Then $704-476=228$; $228\times 2=456$.

The idea that there can be several answers to a problem, not a single answer, is a good one for students to learn early.

RICHARD D. SHIVELY
Mifflin Junior High School
Columbus, Ohio

Dear Editor:

A little practice will enable you to amuse your friends by informing them almost instantly of the correct final digit for any integer raised to an integral power.

Perhaps you will be fortunate enough to be given a number with an exponent divisible by five. It is a well-known number theorem that such a resulting integer will end with the same digit as that of the first power. Because of this, it immediately follows that the sixth, tenth, fourteenth, etc., powers will end with the same digit as does the square of a given integer. In a similar manner the seventh, eleventh, etc., powers will end as does the cube; the fourth power ending will also be the eighth and twelfth.

Since we know our squares and perhaps many of the cubes of the integers, the only drill may be on the fourth powers. But these are just squares squared. Try it.

MERRILL BARNEBEY
University of North Dakota
Grand Forks, North Dakota

Formalization and sterilization

CALEB GATTEGNO, *Binsar, U.K. India.*

The intuitive content of mathematics is as much a reality for learners as for creators.

MATHEMATICIANS are often mistaken for logicians. Among themselves, mathematicians know who is doing what, and often those who produce the meat of mathematics—that is, the new theorems and theories, the promising notions—look upon logicians as the cleaners of the premises, mainly preoccupied with cobwebs. Logicians, in their turn, look down on any creation that has any content; they are convinced that mathematics in essence is one type of game and all the rest is folklore and of lesser value.

This competition among professionals would be an interesting sight for onlookers did it not threaten to invade the field of mathematics education and lead to sterilization in not too distant a future.

If all writers were only grammarians, we can easily imagine how dull bookshops, libraries, and bookshelves in our homes would be. If grammarians believed that only grammar books or articles were literature, we would readily be convinced that illiteracy was around the corner because no one would like to read and no one would find it justified to suggest reading as a source of fun, growth in taste, and experience. In fact, grammar is a codification of what exists already. It provides the specialists with interesting problems that give them excitement and make their efforts worthwhile for themselves. But evolving languages (live, written, and spoken speech) are the justification of grammars and grammarians. Their *raison d'être* is the existence of linguistic challenges that come out of the creative activity of writers and orators.

Similarly for logic and logicians. They certainly have a place in the world of mathematical thinking, but not all the room. They come in, after the mathematician has done some creative work, to look at it and see what they can learn from it or what they can say about it. If mathematicians stopped producing in their way new ideas, soon the activity of the logicians would stop or become purely historical analysis. For the sake of logicians' continued existence, mathematicians must go on thinking in their *artistic* way, i.e., not legislated in advance.

Indeed, mathematicians find their ideas where they can, as they can, and clarify them as far as they can. Their efforts at extracting theorems from the situations they are contemplating cannot be said to follow any one line or even any sequence of lines. Readers of mathematical papers find in them the polished final product of painful and often unsatisfactory struggle. These readers may see that there are still relations to be found in the same situations or in related ones, and, in their turn, produce additional theorems.

The activity of mathematicians is often very complex and has, so far, escaped description. It still remains to find someone who will write an adequate study of mathematical creativity and the actual histories of important contributions in adequate terms. (Poincaré, Hadamard, Pólya, and others have written most interestingly on the subject; still everything seems very mysterious and hidden in this field.) The reasons for such an absence of a comprehensive knowledge on this matter could be: 1) the multiplicity of the approaches of mathematicians to their mental content; 2) the difficulty of creating and at the same time watching how

creation takes place; 3) naïveté mathematicians may experience when doing something other than mathematics, making it difficult to contribute something worthwhile outside the activity for which they trained themselves; 4) the word "mathematics" covers so many areas of experience that it is hopeless to attempt to reduce them to anything but themselves—in other words, mathematics *is* the expression of the activity and must be known from within to be known at all.

If all these reasons were true, to teach mathematics would mean, then, to give experience in as many aspects of mathematics as possible to let the learner find by doing, struggling, and selecting which sections of the wide fields correspond to his gifts, his powers of work, and his insights. Then the acquaintance with real challenges will bear their fruit, and the learner will have the opportunity to become a mathematician who can trust himself to achieve something worthwhile, at least for himself if not for others.

But to teach formalized material is to select for the student; it is to extract all nourishing juice from the situation; it is to present skeletons that only challenge a power of dealing with signs and rules devoid of significance for temperaments different from that of the selectors. Indeed, we fall again into the mistake of teaching grammar instead of literature. Formalization is *one* of the stages of mathematization, and no mathematician can afford to leave out that very important phase (called problem-solving by some) which is the transformation of a perceived situation into a mathematical one—as he would not wish to neglect the presentation of what he has found in the most adequate formal manner current in his day.

To teach mathematics well seems to require that we

1. always stress mathematization of situations either perceived or given by a set of statements, drawings, etc.;
2. relate the schematized new situation with other sets of relations already explored by similarity, analogy, transformations, specialization, or generalization;
3. isolate in the situation one or more relations and their bonds or dynamic links;
4. express most adequately at the level of actual experience of the learners all that has been reached;
5. discuss among the learners, and sometimes with a more advanced student (the teacher), whether the expressions proposed cover the experience met, and check all that in an effort of formalization.

Formalization is thus the last step in the mathematical activity of learners as well as of creators. If we do not place it there, we are courting sterilization, since the meaning of the last stage will have to be guessed on uncertain grounds and lead to a feeling of magic, instead of an activity of knowing minds. It would also unnecessarily tax the imagination that may well be adequate if meaning is always present, but that does not accept guessing at random and without direction.

Teachers of mathematics, as much as any elder members of our community, carefully have to avoid impairing the mental health of the young generation. It is our view that anyone who forces children into a sterile dialogue with meaningless notions and a formal sequence of ideas presented at the beginning of experience, rather than at the final stages of analysis, is actually taking a very grave responsibility in the social sense.

Can we learn music without tunes? Can we learn languages without contexts? Can we learn to swim without being in the water? Why should we then learn mathematics without the very substance that makes it?

These are not emotional notes. They are common sense at a moment when scientific and mathematical education is suddenly beset by professors coming from ivory towers to *tell* what we should be doing with our children—whom they have

perhaps never studied. Our own responsibility begins at the moment we accept the leadership of these men (clearly not resulting from their work in education and with learners, but in far-removed areas of activity). Our responsibility increases if we accept their opinions as true, particularly if they are not.

If the ideas of these professors are right, we shall be grateful. But if these ideas are not correct, have we prepared ourselves to present our own alternatives, or do we only have recourse to another professor?

A man of experience may not be able to value experience until he reflects on how he reached the level of maturity that is his. But teachers, who have looked at pupils learning, know that the children's way of knowing mathematics is through deeper and deeper acquaintance with the labile entities they meet. Children have to know the classes of beings which are covered by names and how to classify into subclasses when attributes are added to the defining ones. As mathematics is, in part, an attitude toward reality in which relations are singled out in their dynamics—half involved in the situation where they are met and half free to be involved in other situations—an important part of mathematical experience is in the acquaintance with that psychological *ambiance* which gives different meanings to the same signs on paper. Just as notes are notes and not tunes, mathematical signs are signs and not mathematics; it is the activity behind them that matters. Every mathematician knows that a mental context goes hand in hand with a mathematical text, and he knows how important it is to be able to tune in when contemplating a situation. All this is indeed folklore, but how important! Without this there may not be any mathematics, as we can observe from the complete lack of any consequence of the axiomatic systems produced in large quantities forty or so years ago. Even when mathematicians say that they play a game with some rules, they put in the game images, meanings, and intuitive relations that guide them in their moves.

If imagery, initiative and sets of meanings are important for the performance of mathematical activity at all levels, we must take great care that our pupils experience them for what they bring while they spend their time at mathematics.

The substance of mathematics in education *is* its folklore, and the more varied it is, the keener our minds will be in finding behind the great variety of experience those large, unifying structures that are like the machine tools of mathematics. Once these structures are perceived, they give a new meaning to all previous experience; but we cannot likewise expect that the contemplation of machine tools may lead everyone, and with reasonable ease, to visualize more structured beings that are useful companions in life.

As teachers, we would like to be successful in the classroom and, if possible, all the time and with most children. That legitimate wish, based upon the knowledge that our pupils have keen minds ready to do and undo complex arrangements and see things happen as a result of their activity, can be reconciled with the fact that for logicians what is most important in mathematics education is to learn to reason according to certain rules. Indeed, we can well see that since there are several years reserved for mathematical education, we may be permitted to build up both a varied and interesting experience in studying the folklore as a source of challenging questions, and the means to find in them the germs of new attitudes that have developed historically to the present. This will also allow us to see that it is possible that the mathematics of tomorrow may be very different from the one fashionable today.

Since the intuitive content of mathematics is as much a reality for learners as for creators, we would be well advised to maintain it in all our reforms, at all levels of teaching. This is one of our jobs as teachers when we examine proposals coming from different horizons.

Constructing logic puzzles

HORACE E. WILLIAMS, *Vanderbilt University, Nashville, Tennessee.*

Here is an interesting and enjoyable method for studying elementary symbolic logic and mathematical reasoning.

IN THE PAST FEW YEARS there has been a trend toward the inclusion of general mathematics courses or courses in modern algebra in the secondary-mathematics curricula. Units in logic and mathematical reasoning are often included in these courses and seem to be popular with the students. Connective statements, truth tables, simple symbolic logic, and mathematical reasoning are topics encountered in these units. The intent of this article is to present a technique for studying logical reasoning that will (1) be appropriate for inclusion in such units; (2) require student participation; (3) be interesting and challenging; and (4) stimulate individual initiative.

The background for this article came from the following argument, adapted from Lewis Carroll:

If he goes to a party, he does not fail to brush his hair. (1)
To look fascinating, it is necessary to be tidy. (2)
If he is an opium eater, then he has no self-command. (3)
If he brushes his hair, he looks fascinating. (4)
He wears white kid gloves only if he goes to a party. (5)
Having no self-command is sufficient to make one look untidy. (6)
Therefore, . . .

We are then asked to supply the valid conclusion to this argument [1, page 46].*

To lay a foundation for the discussion of constructing logic puzzles, let us outline the symbolism and concepts that a student would need to solve the above problem and then show how these may be used to accomplish such a solution and arrive at the conclusion.

A relatively small amount of symbolism and a few basic concepts are all that the student must master to attack the problem. The following items constitute what I consider essential:

1. The student should be introduced to the "If . . . then . . . " conditional statement with its symbolic notation $p \rightarrow q$ (if p then q).
2. He needs to be familiar with the symbolism $\sim p$ for the negation of statement p, and the relationship $\sim(\sim p) \equiv p$ (the double negation of the statement p is equivalent to p).
3. The student should learn how to write conditional statements in symbolic language as follows:
   "p only if q" as $p \rightarrow q$ or $\sim q \rightarrow \sim p$,
   "p if q" as $q \rightarrow p$,
   "p is a sufficient condition for q" as $p \rightarrow q$,
   "p is a necessary condition for q" as $q \rightarrow p$.
4. The student needs to learn the simple syllogism relationship that if $p \rightarrow q$ and $q \rightarrow r$ then $p \rightarrow r$. (This is sometimes referred to as "the chain of logical reasoning.")
5. Finally, the student must be introduced to the contrapositive statement, $\sim q \rightarrow \sim p$, and the concept of the equivalence of a statement and its contrapositive developed ($p \rightarrow q$ is equivalent to $\sim q \rightarrow \sim p$).

With this background, most students are now ready to construct their own logical puzzles. (It seems to me worth

* Numbers in brackets refer to references at the end of the article.

noting at this point that in developing the background we have given the student some basic concepts in logical reasoning.) As a first step, let us assign symbols to the simple statements involved in Lewis Carroll's puzzle:

p = he goes to a party,
q = he brushes his hair,
r = looking fascinating,
s = being tidy,
t = he is an opium eater,
u = he has no self-command,
v = he wears white kid gloves.

Then let us write the six given compound statements in symbolic form.

Statement (1) becomes $p \to q$. (1)′
Statement (2) reworded is "Being tidy is a necessary condition for looking fascinating," which becomes $r \to s$. (2)′
Statement (3) becomes $t \to u$. (3)′
Statement (4) becomes $q \to r$. (4)′
Statement (5) becomes $v \to p$. (5)′
Statement (6) becomes $u \to \sim s$. (6)′

Next let us piece the symbolic statements together into syllogisms and seek the valid conclusion to the argument. Starting with statement (1)′, $p \to q$, and combining it with statements (4)′ and (5)′ we get,

$$v \to p \to q \to r. \qquad (7)$$

Then combining (7) with statement (2)′ and the contrapositive of statement (6)′, $[\sim(\sim s) \to \sim u$ or $s \to \sim u]$, we get,

$$v \to p \to q \to r \to s \to \sim u. \qquad (8)$$

Finally, combining (8) with the contrapositive of statement (3)′, $(\sim u \to \sim t)$, we deduce,

$$v \to p \to q \to r \to s \to \sim u \to \sim t. \qquad (9)$$

The valid conclusion to this syllogism then becomes

$$v \to \sim t,$$

which is the statement, "If he wears white kid gloves then he is not an opium eater," when translated from symbolic language to written language.

After having introduced the necessary symbolism and concepts of logic and having used these to unscramble and solve a logic puzzle, such as the one given, the student is now ready to construct logic puzzles of his own.

At this point, a good consolidating educational technique for the teacher would be to have the students construct their own logic puzzles, bring them to class, and have other students attempt solutions. Perhaps it would be beneficial to encourage the students to interject some nonsense and humor into their syllogisms to add to the enjoyment. They should first strive to construct simple puzzles, and then, as their facility improves, be encouraged to construct more complicated ones.

For illustrative purposes, I have included two sample puzzles, showing the steps involved in their construction. The first is a simple one involving four conditional statements, while the second contains seven conditional statements and is a bit more complicated.

Example I

Step 1. Make up a syllogism using straightforward if . . . then . . . language, keeping the statements in order and stating the valid conclusion to the argument.

If it rains on Saturday, then there will be no dance.
If there is no dance, then I will go to the movie.
If I go to the movie, then I will have no money left.
If I have no money left, then I can read no new comic books.
Conclusion: If it rains on Saturday then I can read no new comic books.

Step 2. Complete the puzzle by rearranging the wording of the statements and mixing up their order of presentation.

If I go to the movie, I will have no money left.
There is no dance only if I go to the movie.
Rain on Saturday is sufficient for there to be no dance.

Reading no new comic books is a necessary condition for having no money left.

Example II

Step 1.

If it frosts in September, then the pears will not grow.

If the pears will not grow, then the children wear old shoes.

If the children wear old shoes, then they wear holes in their socks.

If the children have holes in their socks, then the sock business booms.

If the sock business booms, then Joe buys a new automobile.

If Joe buys a new automobile, then he has dates quite often.

If Joe has dates quite often, then all the young ladies are happy.

Conclusion: If it frosts in September, then all the young ladies are happy.

Step 2.

The children wear holes in their socks if they wear old shoes.

The sock business booming is sufficient for Joe to buy a new car.

The children will not wear old shoes, only if the pears grow.

For the children to wear holes in their socks, it is necessary for the sock business to boom.

If it frosts in September, the pears do not grow.

All the young ladies being happy is a necessary condition that Joe have dates quite often.

Joe does not have dates quite often, only if he does not buy a new car.

The student now brings the final statement of the puzzle from step 2 and presents it in class for other members of the class to solve. Healthy and lively competition among the students may develop in trying to solve one another's puzzles and should be encouraged, particularly in ability-grouped classes where students are competing on homogeneous aptitude levels.

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3 EVES, HOWARD, and NEWSOM, CARROLL V. *An Introduction to the Foundations and Fundamental Concepts of Mathematics.* New York: Rinehart and Company, 1958.

Have you read?

RISING, GERALD R. "Some Comments on Teaching of the Calculus in Secondary Schools," *The American Mathematical Monthly,* March 1961, pp. 287–290.

There seems to be a big question on the place of calculus in the secondary school. The author of this article as a teacher of calculus in the secondary school and the university presents some pertinent points of view.

For example, he maintains that the very best high school teachers are assigned to calculus classes in comparison to graduate assistants who are assigned to many colleges classes. He feels the CEEB exams are sophisticated and anyone who passes the examination certainly has competent understanding. He feels that accelerated mathematics students do know basic principles and should have the opportunity to do advanced work.

He admits there are weak teachers, weak programs, and students in the programs who are not prepared, but this is not representative of the whole program. You will be interested in the description of several programs.

Mr. Rising offers such sound recommendations to college critics of secondary teaching as these: communicate directly to the school where the poorly prepared students graduated, compare criticisms with those made by graduate schools about your colleges, and realize that college programs may need changing.

This is a good article for all of us to read and then draw conclusions.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Mathematics projects

SISTER MARGARET CECILIA, C.S.J.,[1] *Salpointe High School, Tucson, Arizona.*

Mathematics teachers can use a variety of approaches to develop interest and understanding. The use of student projects can lead to independence and inventiveness in mathematical thought

A MATHEMATICS PROJECT should be designed for developing greater insight, interest, and motivation in mathematics. The present concern of teachers for the revision of the curriculum and improvement of the teaching of mathematics can find expression in the form of experiments with mathematics projects. These projects should aim to enrich the pupil's entire learning process in mathematical pursuits and not merely be the addition of extra work for the pupils concerned; projects should be used as a supplementary activity to regular classroom work or in mathematic clubs and other activities not necessarily connected directly with the work in the classroom. Science projects, in general, should be used for furthering pupil interest and development in specific areas. In the area of mathematics, the teacher can hope to extend the student's knowledge and mathematical power. For the student, there should be a deepening and strengthening of understanding, and this, in turn, should lead to independence and inventiveness in mathematical work.

How does one begin to interest students in a project?

In the beginning of the school year, the teacher should discuss topics with the pupils; possibilities should be considered by teacher and students. After lists of topics and a bibliography have been given to the pupils, the teacher should encourage each student to select a topic as soon as possible. This topic should then be approved by the teacher. This means individual guidance—advice as to feasibility, extent of research, available materials, expense (this should be kept at a minimum—one of our prize-winning projects in mathematics submitted in the Westinghouse Talent Search cost a total of $2.50), construction, and so on. As an incentive for the students, projects can be entered in the school science fair, local fair, or the National Science Fair—if such be your good fortune. At Salpointe High School, we have been fortunate enough to have three winning mathematics projects at the Southern Arizona Science Fair, and these projects represented our area in the National Science Fairs of 1957, 1958, and 1961.[2] This was indeed an honor, since these projects competed with projects in such areas as chemistry, physics, and engineering. Our experiences are cited here by way of encouragement and motivation for you as a teacher. Pupils can do projects that are really worth while. Not only can this be a learning experience for the students, but it is often a tremendous learning experience for the teacher—this was certainly true in my case. If a science fair is

[1] Sisters of St. Joseph of Carondelet, Los Angeles Province.

[2] In 1957 the National Science Fair was held in Los Angeles, California, in 1958 in Flint, Michigan, and in 1961 in Kansas City, Missouri. On these occasions, the student and I received all-expense-paid trips. Richard Jensen, who submitted a mathematics project in the Westinghouse Talent Search received an all-expense-paid trip to Washington, D.C.

not held in your community, the mathematics projects can be displayed at Open House for the parents or at some other school function. In any case, I think it is very important that students receive some recognition for their achievements in the realm of mathematics—awards, for example, could be given at a student assembly.

What does the mathematics project demand of the teacher?

Success in the development of student interest in mathematics projects depends in a large measure on the teacher's ability to motivate pupils to undertake topics for study. The teacher himself must be interested, enthusiastic, and ready to give of himself in the way of time and individual guidance. This, of course, means hard work. There is no "royal road" to the development of good mathematics projects. However, this does not mean that the teacher is to do the work of the student; it is the teacher's business to provide a list of topics from which the students may derive ideas, to discuss these topics with the students, to guide the individual student in the development of his project, and to provide adequate resource materials.

Literature is an essential part in the research of a mathematics project. Pupils must be made aware of the literature available. Bibliographies should be distributed and many pupils will read quite widely on topics which otherwise might have escaped their attention. It is, therefore, important that a good selection of mathematics books be available in your school for student use. It is also advisable, if this be possible in your particular school management, to keep these books in the department of mathematics rather than in the school library. In this way, the mathematics teachers will be able to control book circulation. During the period of research, these books are in great demand, and it is often necessary that books be checked out for only a week or a few days.

How can you discover new ideas for projects?

At the end of this article you will find a list of mathematics projects which have been done by students at our school. The list is by no means complete. Time is at a premium today and, for the teacher, the task of locating and collecting appropriate topics for research in mathematics can be formidable. With this in mind, I have compiled a list of topics for mathematics projects. The list is the result of many hours of "book browsing"; these topics have been collected over a period of several years. I might add that many of the articles in THE MATHEMATICS TEACHER have proved most helpful, and in many instances these articles formed the beginnings of new ideas.

At our school we have had as many as thirty projects in mathematics entered in one school science fair.[3] Some of these projects were on display in 1960 at the Arizona Mathematics Teachers meeting. The mathematics projects of last year were a decided improvement over those of previous years. Each year they seem to get better. I think you will find this true of all science projects; after several years of experience the projects begin to take on a professional appearance.

It is my sincere hope that the following list of projects and the preceding remarks will prove helpful to you as a mathematics teacher. Much of our modern science today is carried on by ordinary people who, perhaps unconsciously motivated by the enthusiasm of a teacher, started out as high school students interested in mathematics and science.

One hundred topics for mathematics projects

Finding the Center of Gravity
Studying the Twist and Turn of a Third-Dimensional Curve by Use of the Osculating Plane

[3] This was about 37½ per cent of the number of students enrolled in my mathematics classes, excluding general mathematics.

Napier's Rods
The First Twelve-Point Sphere of an Orthogonal Tetrahedron
Minimal Surfaces
Investigating the Cycloid
Geometry in Practical Use
Projective Geometry
Mosaics by Reflection
Prime Numbers and the Sieve of Eratosthenes
Projectiles
Satellites—What Keeps Them Up
Investigation of the Fourth Dimension
What Is a Spiral?
Lobachevskian Geometry
What Is a Group?
Pascal's Theorem
Desargue's Configuration
Knots
One-Sided Surfaces
Linkages
Repeated Reflections
The Tetrahedron Tower
Geometry and Graphics
Topology and Reasons
Symmetry and the Fourth Dimension
Magic Squares
Mathematical Principles of Particle Acceleration
Dürer's Magic Square
Linkages in Third Dimension
Mathematics of Crystals
The Reflex Arc
Geometrical and Relativity Concepts of Four Dimensions
Ruled Surfaces
Nomographs—Solving Simultaneous Equations
Investigating the Nine-Point Circle in Three-Space
Trigonometric Functions—the Unit Circle
The Magic Knight's Tour (Chess)
The Four-Color Problem
Color Problems in Cartography
Binomial Notation
The Snowflake
The Number *e*
The Hyperbolic Paraboloid
The Three-Hundred-Year Calendar
Digital Conversion
Pendulum Designs
Pappus' Extension of the Pythagorean Theorem
The How of Computers
Fun with Hexaflexagons
Time Curves
World's Oldest Adding Machine and How It Works—the Abacus
Quadric Surfaces
The Map Coloring Problem by Use of Networks
Geometric Psychic Diagnosis
Brocard Points in Aviation
Investigating the Binomial Theorem
Group Theory of the Equilateral Triangle
Tracing the Cardioid
Evaluation of Pi
Regular Polygons
The Error Curve in Modern Science Mathematics
Mathematics in Roulette
Mathematical Wing Structure of the Pear Thrips
Geometric Transformations
From Sticks to Numerals
The Golden Ratio of Phi
What's in a Braid?
Mathematics in Gears
Mathematics in Nature
The Transit
How To Lie with Statistics
Soap Bubbles—the Question of Curvature
Bubble Curves and the Roulettes of Conic Sections
Why It Sparkles
Mathematics in Photography
Mathematics in Music
The Möbius Strip
Geometric Dissections—Tangrams
The Regular Seventeen-Sided Polygon
The Nine-Point Circle
The Euler Line
The Quadratic Slide Rule
Probability Determination of Pi
Paper-Folding
Perfect Numbers
Geometric Solutions of Quadratics
Parabolic Sound Reflector
The Highway Clover Leaf
Optical Illusions

The Riddle of the Slide Rule
Determining the Volume of an Irregular Solid
Game Theory
The Kaleidoscope
Boolean Algebra
De Moivre's Theorem
Riemannian Geometry
Geometric Extraction of Cube Roots
The Law of Growth
Number Theory

What's new?

BOOKS

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Arithmetic for College Students, L. J. Adams. New York: Holt, Rinehart, & Winston, 1961. Cloth, ix+262 pp., $3.75.

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Nomography

C. R. WYLIE, JR., *University of Utah, Salt Lake City, Utah.*
An interesting application of several important topics from algebra and analytic geometry offers the students an opportunity to exercise their ingenuity.

SINCE THE DEVELOPMENT of nomography toward the end of the nineteenth century, largely by the French mathematician Maurice d'Ocagne, the art of nomography has been cultivated chiefly by engineers. This is natural, of course, for nomograms, or alignment charts as they are more descriptively called, are convenient devices for the rapid evaluation of numerous formulas in three and sometimes more variables where a high degree of accuracy is not required. It seems unfortunate, however, that a comparable interest has not been exhibited by teachers of mathematics, for nomography provides an interesting application of several important topics from algebra and analytic geometry and abounds with well-motivated problems offering ample opportunity for the exercise of ingenuity.

The present paper is primarily expository, although several of the nomograms developed as illustrations, if not actually new, at least seem but little known and should be of interest to teachers and students of algebra and trigonometry.

PRELIMINARY NOTIONS

In its simplest form, a nomogram for a formula $\phi(u, v, w)=0$ is an array of three scales, either straight or curved, one for the variable u, one for v, and one for w, with the property that if u_1, v_1, and w_1 are values such that $\phi(u_1, v_1, w_1)=0$, then the points determined on the respective scales by the values u_1, v_1, and w_1 are collinear, and conversely. The utility of such a chart is obvious, for if any two of the variables, say u_1 and v_1, are given, then the value(s) of the third variable, w, which corresponds to them can be found without solving the equation $\phi(u_1, v_1, w)=0$ simply by drawing a straight line through the points determined on the u- and v-scales by the values u_1 and v_1 and observing the value(s) of w associated with the point(s) where this line crosses the w-scale. It is interesting that for the construction of charts with this elegant property all that we require are a few simple results from algebra and analytic geometry.

In the first place, we need to know that if $P_1:(x_1, y_1)$, $P_2:(x_2, y_2)$, and $P_3:(x_3, y_3)$ are any three points in the Cartesian plane, these points will be collinear if and only if

$$(1)\qquad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Next we must recall that any multiple of the elements of any column of a determinant may be added to the corresponding elements of any other column without changing the value of the determinant. Moreover, if the elements of any row of a determinant are multiplied or divided by any (nonzero) quantity, the determinant is multiplied or divided by that quantity. Specifically, if the value of a determinant is zero, these operations leave its value zero.

Finally, we must use the fact that curves may be described in Cartesian coordinates not only by equations of the form $y=f(x)$ and $F(x, y)=0$ but also by

parametric equations, say $x=x(t)$ and $y=y(t)$. This mode of representation is of course familiar to us from analytic geometry, but we shall have to extend it slightly. In the usual applications, a curve defined parametrically is plotted by assigning a series of values to the parameter t, computing the corresponding values of x and y, and then locating the points whose coordinates are the pairs of associated x and y values *without regard to the values of* t *which determined them.* For our purposes, as we shall soon see, it is essential that we remember, and in fact record on the curve, the value of the parameter associated with each point we plot. This means that each curve we draw will bear a scale (like the scales of a slide rule, for instance) by means of which the correspondence between points on the curve and values of the parameter will always be evident.

The construction of a nomogram

Suppose that we are given the formula $\phi(u, v, w)=0$ and that by appropriate manipulations we are able to rewrite it as a determinantal equation,

$$(2) \qquad \begin{vmatrix} f_1(u) & f_2(u) & 1 \\ g_1(v) & g_2(v) & 1 \\ h_1(w) & h_2(w) & 1 \end{vmatrix} = 0$$

with the following characteristics:

a) Each row of the determinant contains one and only one of the unknowns, and
b) The third column of the determinant consists exclusively of 1's.

Not all equations in three unknowns can be so rearranged, but when it is possible, it can be done in infinitely many ways.

Now to obtain a nomogram for the formula $\phi(u, v, w)=0$ we have only to construct the three curves whose parametric equations are

$$\begin{array}{lll} C_u: & x=f_1(u) & y=f_2(u) \\ C_v: & x=g_1(v) & y=g_2(v) \\ C_w: & x=h_1(w) & y=h_2(w) \end{array}$$

marking each with the scale of the corresponding parameter.[1] For if (and only if) the points corresponding to $u=u_1$, $v=v_1$, and $w=w_1$ are collinear, then their coordinates, namely

$$[f_1(u_1), f_2(u_1)], \qquad [g_1(v_1), g_2(v_1)], \qquad [h_1(w_1), h_2(w_1)]$$

will satisfy (1), and therefore from (2) it follows that $\phi(u_1, v_1, w_1)=0$, as required.

It is now clear why we must insist on conditions a) and b). For if the last column of the determinant in (2) did not consist exclusively of 1's, then (2) would not have the form of the collinearity condition, (1), as our argument requires. Also, if more than one variable appeared in any row of the determinant in (2), then the first two elements in that row would involve more than one parameter and hence would not constitute the parametric representation of any curve, much less the scale curve for any one of the unknowns.

Nomograms for addition

The simplest formula to represent nomographically is undoubtedly the addition formula

$$w=u+v.$$

As a first step in the construction of a nomogram for this relation, we attempt to rewrite it as a determinantal equation in which each row of the determinant involves one and only one of the variables. This is easily done by inspection,[2] and we obtain at once, among many other possibilities,

$$(3) \qquad \begin{vmatrix} u & 1 & 0 \\ v & 0 & 1 \\ w & 1 & 1 \end{vmatrix} = 0.$$

[1] Clearly, all that we need in a nomogram are the scales of the three variables. Hence the x- and y-axes, which of course must be used in plotting the scale curves, are never shown in the final chart.

[2] In most elementary problems the initial step of writing the given equation in determinantal form can be accomplished by means of the identity

$$h_1(w)-f_1(u)h_2(w)-g_1(v)h_3(w) \equiv \begin{vmatrix} f_1(u) & 1 & 0 \\ g_1(v) & 0 & 1 \\ h_1(w) & h_2(w) & h_3(w) \end{vmatrix}$$

While this obviously meets condition a) of the last section, it does not meet condition b). Hence, using the laws of determinants, we must modify its form until we obtain one in which only 1's appear in the last column. Moreover, in doing this we must be careful to use column operations only and not row operations. Combining two or more rows will lead to a form in which at least one row will contain two or more of the variables, in violation of condition a).

The required manipulations can be carried out in infinitely many ways. For instance, we may add the second column to the third and then divide each element of the last row by 2, getting

$$\begin{vmatrix} u & 1 & 1 \\ v & 0 & 1 \\ \frac{w}{2} & \frac{1}{2} & 1 \end{vmatrix} = 0.$$

This now meets each of our requirements. The variable u is represented by a uniform scale on the line $y=1$; v is represented by a uniform scale on the line $y=0$; and w is represented by a uniform scale on the line $y=\frac{1}{2}$. The general appearance of the nomogram is shown in Figure 1a.

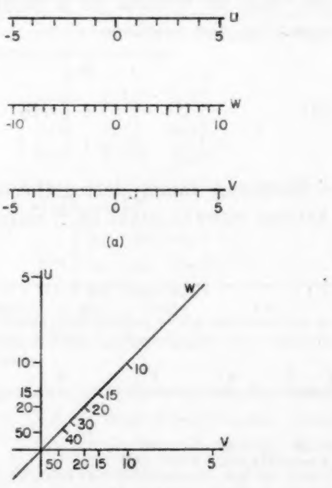


Figure 1 Nomograms for $w=u+v$

On the other hand, we could equally well have divided the elements of the first row of (3) by u, the elements of the second row by v, and the elements of the third row by w, and then interchanged the first and third columns, getting

$$\begin{vmatrix} 0 & \frac{1}{u} & 1 \\ \frac{1}{v} & 0 & 1 \\ \frac{1}{w} & \frac{1}{w} & 1 \end{vmatrix} = 0.$$

This is equivalent to the original equation, $w=u+v$, provided $uvw \neq 0$, and clearly satisfies both condition a) and condition b). In this case, u, v, and w are represented by reciprocally graduated scales on the lines $x=0$, $y=0$, and $y=x$, respectively. The corresponding nomogram is sketched, with a skeleton of values, in Figure 1b.

It is interesting to note that the nomogram of Figure 1a is especially useful if relatively small values of u, v, and w are involved and is of little use if very large values must be handled. On the other hand, the nomogram of Figure 1b is of little or no use for very small values of u, v, and w, but is well adapted to calculations involving larger values.

This is typical of the general situation. Since no chart can extend indefinitely, it follows that except in those relatively rare cases where each of the scales lies on a curve without infinite branches, some portion of at least one of the scales must fall outside the limits of the actual drawing. Just which portions of which scales are inaccessible depends upon the form of the determinant from which the nomogram is constructed, that is, upon the set of manipulations by which the standard form (2) is achieved. At the elementary level of our discussion this is largely a matter of trial and error, disciplined by ingenuity and imagination. At a somewhat more sophisticated level, the theory of linear transformations in the projective

plane makes possible a systematic development of nomograms in which, within wide limits, predetermined sections of the scales of the variables can be made accessible and others relegated to the inaccessible region beyond the boundaries of the chart.

The formula $w=u+v$ is so simple to evaluate that there is no reason for creating a nomogram to expedite the process. However, when the individual terms are more general functions than the variables themselves, that is, when the sum is of the form

$$H(w)=F(u)+G(v),$$

the problem is not trivial, and a nomogram for it is often convenient. As an example of this sort, consider the quadratic equation

$$t^2+bt+c=0$$

whose solution, of course, is

$$t=-\frac{b}{2}\pm R \quad \text{where} \quad R^2=\frac{b^2-4c}{4}\cdot$$

Clearly, the determination of t can be made to depend upon two addition-type nomograms, one to compute R when b and c are given and one to compute t from b and R. The second addition is so simple that a nomogram is really unnecessary. The calculation of R, however, can be greatly facilitated by an appropriate chart.

Now it is easy to verify that $R^2+c-b^2/4=0$ can be written as the determinantal equation

$$\begin{vmatrix} R^2 & 0 & 1 \\ -2c & 1 & 1 \\ \dfrac{b^2}{2} & -1 & 1 \end{vmatrix}=0.$$

This is in standard form, and it is a simple matter to construct the scales for each of the variables. For R we have a nonuniform scale on the line $y=0$; for c we have a uniform scale on the line $y=1$; and for b we have a nonuniform scale on the line $y=-1$. The complete chart is shown in Figure 2. When R^2 is negative, R is imaginary, as indicated on the R-scale. Hence our nomogram, coupled with the auxiliary formula

$$t=-\frac{b}{2}\pm R$$

allows us to solve a quadratic equation when its roots are complex as well as when its roots are real. In passing, we note that by taking $b=0$, square roots and squares can be read from Figure 2.

Nomograms for multiplication

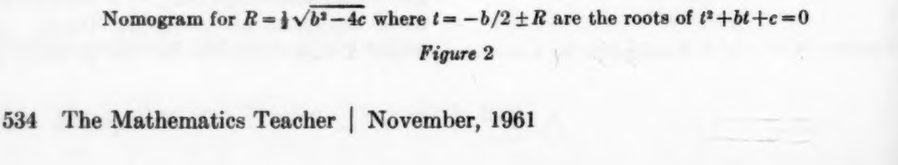
The formula

$$w=uv$$

can also be written as a determinantal equation, for instance

$$(4) \qquad \begin{vmatrix} u & 1 & 0 \\ 0 & v & 1 \\ w & 0 & -1 \end{vmatrix}=0.$$

Hence by manipulating the columns in various ways to make each element in the



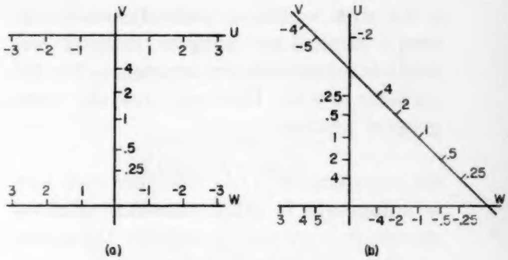
Nomogram for $R=\frac{1}{2}\sqrt{b^2-4c}$ where $t=-b/2\pm R$ are the roots of $t^2+bt+c=0$

Figure 2

last column equal to 1, we can obtain any number of nomograms for multiplication.[3] Specifically, by adding k times the second column to the third and then dividing the elements of the first row by k, the elements of the second row by $(1+kv)$, and the elements of the third row by -1, we have

$$\begin{vmatrix} \dfrac{u}{k} & \dfrac{1}{k} & 1 \\ 0 & \dfrac{v}{1+kv} & 1 \\ -w & 0 & 1 \end{vmatrix} = 0.$$

This is in the standard form (2), and therefore can be used for the construction of an alignment chart. It is interesting to note that on the v-scale (the line $x=0$) values in the neighborhood of $v=-1/k$ are inaccessible. By choosing k suitably, we can therefore relegate the neighborhood of any value of v, except $v=0$, to the inaccessible region outside the chart. Figure 3a shows a sketch of the nomogram for $k=1$.



Nomograms for $w=uv$

Figure 3

On the other hand, we might first multiply the elements in the second column of (4) by k and the elements in the third column by l and add them both to the corresponding elements in the first column. Then by dividing the elements in each row by the first element in that row and finally interchanging the first and third columns, we obtain

$$\begin{vmatrix} 0 & \dfrac{1}{k+u} & 1 \\ \dfrac{1}{l+kv} & \dfrac{v}{l+kv} & 1 \\ \dfrac{1}{l-w} & 0 & 1 \end{vmatrix} = 0.$$

In the nomogram based on this determinant, the scale of u appears on the line $x=0$, with values around $u=-k$ inaccessible; the scale of v appears on the line $lx+ky=1$, with values around $v=-l/k$ inaccessible; and the scale of w appears on the line $y=0$, with values around $w=l$ inaccessible.[4] Figure 3b shows a sketch of this nomogram in the simple case $k=l=1$.

[3] Alternatively, of course, nomograms can be obtained by the methods of the last section by first taking logarithms and writing $w=uv$ in the form $\log w = \log u + \log v$.

[4] The most general procedure for introducing arbitrary constants, which can subsequently be chosen to achieve desired characteristics in a nomogram, is based upon the following considerations:
Let

$$N = \begin{vmatrix} f_1(u) & f_2(u) & f_3(u) \\ g_1(v) & g_2(v) & g_3(v) \\ h_1(w) & h_2(w) & h_3(w) \end{vmatrix} = 0$$

be one determinantal equation equivalent to a given formula $\phi(u, v, w)=0$, and let

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

be a nonvanishing determinant whose elements are arbitrary constants. Then by the multiplication theorem for determinants

$$NA = \begin{vmatrix} (a_{11}f_1+a_{21}f_2+a_{31}f_3) & (a_{12}f_1+a_{22}f_2+a_{32}f_3) & (a_{13}f_1+a_{23}f_2+a_{33}f_3) \\ (a_{11}g_1+a_{21}g_2+a_{31}g_3) & (a_{12}g_1+a_{22}g_2+a_{32}g_3) & (a_{13}g_1+a_{23}g_2+a_{33}g_3) \\ (a_{11}h_1+a_{21}h_2+a_{31}h_3) & (a_{12}h_1+a_{22}h_2+a_{32}h_3) & (a_{13}h_1+a_{23}h_2+a_{33}h_3) \end{vmatrix}$$

and, since $A\neq 0$, the last determinant will be zero if and only if $N=0$, that is, if and only if $\phi(u, v, w)=0$. For arbitrary values of the a_{ij}'s, provided only that $A\neq 0$, the last determinant can therefore be used to construct a nomogram for $\phi(u, v, w)=0$ after its third column has been reduced to 1's. This is the algebraic equivalent of the use of projective transformations which we mentioned in an earlier footnote.

As with addition, multiplication is so simple that there is little practical reason for constructing a nomogram for the formula $w=uv$. However, for the more general relation

$$H(w)=F(u)G(v)$$

a nomogram is often desirable. For instance, it is natural to consider the nomographic representation of the trigonometric formulas

$$b=h\sin\theta,\qquad a=h\cos\theta,\qquad b=a\tan\theta.$$

As always, there are infinitely many possibilities, no one of which is "best." For $b=h\sin\theta$ one particularly useful one, based on the determinantal representation

$$\begin{vmatrix} -1 & \dfrac{7h}{5} & 1 \\ 0 & b & 1 \\ \dfrac{\sin\theta}{\dfrac{7}{5}-\sin\theta} & 0 & 1 \end{vmatrix}=0$$

is shown in Figure 4. Moreover, since $a=h\cos\theta$ and $b=h\sin\theta$ are identical in structure, and since $\sin\theta=\cos(\pi/2-\theta)$, this nomogram can also be made to serve for $a=h\cos\theta$ simply by labeling each point on the θ-scale with the complementary value $\pi/2-\theta$ and allowing the scale on the line $x=0$ to represent a as well as b. In passing, we note that by taking $h=1$ the values of $\sin\theta$ and $\cos\theta$ can be read directly from the nomogram of Figure 4, since in this case we have simply $b=\sin\theta$ and $a=\cos\theta$.

To complete our nomographic "tools" for the solution of right triangles we should have a nomogram for the third formula $b=a\tan\theta$. The construction of such a chart follows closely the pattern we have just described, however, and we shall leave the details to the interested reader (or his students!).

Additional topics

The examples we have so far given have all involved nomograms in which the scale of each variable is a straight line. This is not always the case, and for many formulas one or more of the variables will have a curved scale. For instance, if we rewrite the quadratic equation $t^2+bt+c=0$ in the initial form

$$\begin{vmatrix} t^2 & t & 1 \\ b & -1 & 0 \\ c & 0 & -1 \end{vmatrix}=0,$$

it is an easy matter to achieve the standard form

$$\begin{vmatrix} \dfrac{t^2}{1+t^2} & \dfrac{t}{1+t^2} & 1 \\ 1 & -\dfrac{1}{b} & 1 \\ \dfrac{c}{c-1} & 0 & 1 \end{vmatrix}=0.$$

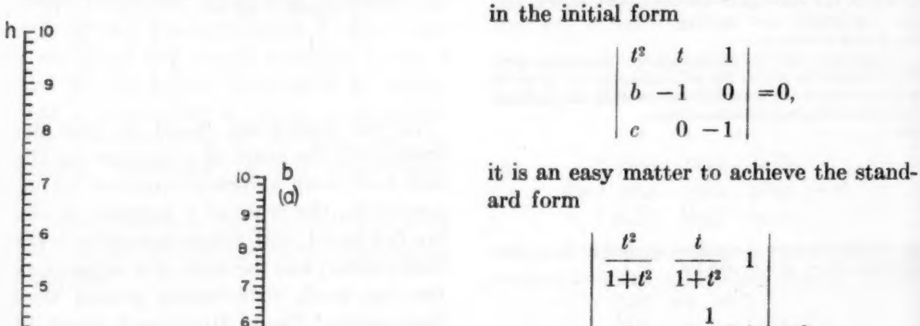


Figure 4 Nomogram for the evaluation of $b=h\sin\theta$ and $a=h\cos\theta$

Here the scale of c lies on the line $y=0$, the scale of b lies on the line $x=1$, and the scale of t lies on the curve whose parametric equations are

$$x=\frac{t^2}{1+t^2},\qquad y=\frac{t}{1+t^2},$$

i.e., the circle $(x-\frac{1}{2})^2+y^2=\frac{1}{4}$. This nomogram is well known, but since it does not give the complex roots of the quadratic it is probably less useful than the one we presented above. It can be found on page 129 of *The Calculus of Observations* by Whittaker and Robinson.

Another interesting formula whose nomographic representation involves a curved scale is

$$a\cos\theta+b\sin\theta=1.$$

Since this can be written in the form

$$\begin{vmatrix} \frac{1}{a} & 0 & 1 \\ 0 & \frac{1}{b} & 1 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}=0,$$

it is clear that a and b appear as reciprocally graduated scales on the x- and y-axes, respectively, while θ appears as a uniformly graduated scale on the circle $x^2+y^2=1$.

The nomographic representation of formulas involving more than three variables, when possible at all, is usually quite intricate. Because of the difficulty of plotting loci in more than two dimensions, the well-known extension of (1) which asserts that four points in a space of three dimensions lie in the same plane if and only if their co-ordinates satisfy the equation

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}=0$$

is obviously of no practical value. The usual procedure is to attempt to rewrite the given equation, say $\phi(u, v, w, z)=0$ as an equality between two functions each involving just two of the four variables, say

$$\phi_1(u, v)=\phi_2(w, z)$$

and then construct the nomograms for the two relations

$$\phi_1(u, v)=t \quad\text{and}\quad \phi_2(w, z)=t$$

where t is the common value of the functions ϕ_1 and ϕ_2. Then given three of the four unknowns, say u, v, and w, we can use the first nomogram to find the value of t corresponding to the given values of u and v. With this and w we can then use the second nomogram to find the value of the fourth variable, z. In many cases the two nomograms can be so arranged that the scale of t is the same in each. When this is possible the line bearing the t-scale need not be graduated, since only the appropriate point, and not the actual value of t, is employed in passing from one nomogram to the other. In particular, by using this technique it is possible to construct nomograms for addition and multiplication relations, such as

$$f_1(z)=f_2(u)+f_3(v)+f_4(w) \quad\text{and}$$
$$f_1(z)=f_2(u)f_3(v)f_4(w)$$

which involve four (or even more) variables.

Conclusion

Nomography, while not an important branch of advanced or "modern" mathematics, nevertheless presents many features which make it an attractive illustration of important topics in algebra and analytic geometry. Formulas which it is capable of handling abound in elementary mathematics, physics, and chemistry. And perhaps best of all, with or without a knowledge of projective transformations, the construction of a nomogram which in some sense will be an "optimum" representation of a formula for specified ranges of the variables calls for considerable ingenuity and imagination.

The additive inverse in elementary algebra

JOEL J. RHEINS, *James Fenimore Cooper Junior High School, New York, New York, and*
GLADYS B. RHEINS, *Great Neck North Junior High School, Great Neck, New York.*

Some ways a teacher may wish to use the field properties.

THE CONCEPT of the additive inverse can be fruitful in the development of the rules for handling signed numbers in addition and in multiplication in elementary algebra. The rationalization of the processes has heretofore made use of analogies and examples of artificial mathematical situations. The use of the additive inverse enables us to remain within the field of algebra and to build the structure on a logical basis.

Zero plays a central role. Pupils are reminded that zero added to any number, as in arithmetic, yields the original number. In this way they are introduced to zero as the identity element in addition. This step reinforces the concept of zero and leads to the question whether the addition of any two numbers could yield zero. The translation of this query into the statement that $a+a'=0$ (a' is read "a prime") leads directly to the notion of the negative number; and a' (a negative number) is denoted the additive inverse of a (a positive number). By extension of this idea, a (a positive number) is the additive inverse of a' (a negative number). Similarly, a may be a negative number and a' a positive number and the same relationships hold.

Pupils at this stage are now ready to apply the idea to develop the rules for determining the sign of the sum in the addition of signed numbers with unlike signs. Addition of numbers with like signs presents no difficulty in ready acceptance on the part of the pupils and this is taken as a starting point. The understanding of the rule for the addition of two numbers of unlike signs is gained readily by generalization from examples, such as

1) $+10+(-7)=[+3+7]+(-7)$
$=+3+[+7+(-7)]$
By the associative law
$=+3+0$
$=+3$
By addition identity

2) $(-10)+(+7)=[(-3)+(-7)]+(+7)$
$=-3+[(-7)+(+7)]$
$=-3$

The pupils discover after more experiences like the two examples above that the sum in such cases is the difference of the absolute values of the two numbers prefixed by the sign of the number having the greater absolute value.

The concept of additive inverse at this stage can be used to reduce the usual two cases of solution of simple equations to one case. For example,

1) $$\begin{array}{r} x+7=10 \\ -7=-7 \\ \hline x=3 \end{array}$$ and 2) $$\begin{array}{r} x-4=3 \\ +4=+4 \\ \hline x=7 \end{array}$$

can be reduced to the single case:

$$
\begin{aligned}
x+a &= b\\
a' &= a'\\
\hline
x+a+a' &= b+a'\\
x &= b+a'.
\end{aligned}
$$

(a and b may be any real numbers, positive, negative, or zero).

The development of the laws for the sign of the product of two numbers is a natural outcome of the previous development. The familiar notion is again recalled that the product of any number and zero yields zero. From experience with arithmetic, the pupils readily agree that the product of two positive numbers is a positive number; i.e., for a and b positive $(a)(b)=(ab)$ and ab is positive.

In the discussion which follows, it will be assumed that a and b are positive numbers unless prefixed by a negative $(-)$ sign. In the latter cases $-a$ and $-b$ will be negative numbers. The case of the product of numbers with unlike signs can be developed by the following formulation:

$$a+(-a)=0$$

$$b=b$$

By identity

$$b[a+(-a)]=b(0)$$

Multiplication of equals axiom

$$ba+b(-a)=0$$

Distributive law

or $\quad ab+(-a)(b)=0$

Commutative law.

Therefore b times $-a$ (or $-a$ times b) must yield the additive inverse of the positive number ab; that is $(-a)(b)$ must be a negative number $-(ab)$ for only thus can $ab+(-ab)=0$ since the additive inverse of a positive number is a negative number. This establishes that the product of two numbers with unlike signs is negative.

Lastly, we develop the idea that the product of two negative factors is positive.

$$
\begin{aligned}
b+(-b)&=0\\
-a&=-a\\
-a[b+(-b)]&=-a(0)\\
(-a)(b)+(-a)(-b)&=0\\
-ab+(-a)(-b)&=0
\end{aligned}
$$

since we already have shown that

$$(-a)(b) \text{ equals } -ab.$$

We know that a number and its additive inverse yields zero. Therefore $(-a)(-b)$ must be the additive inverse of $-ab$. Since the additive inverse of a negative number is a positive number of equal absolute value, $(-a)(-b)$ must be equal to $+ab$.

The teacher who intends to use the concept of the additive inverse and the other field properties for the purposes described above must be careful to be consistent in the terminology employed throughout the course and must plan carefully to insert the development of the concepts at the proper places so that the discoveries seem natural to the pupils. If our experience with this approach is an indication, the extra effort in planning will be found greatly rewarding in the achievement and understanding of the pupils for whose benefit the course is designed.

"Thousands of seniors in high schools with small enrollments are now able to take advanced courses in science and mathematics for the first time."—*From "Schools in Our Democracy," Office of Education, U.S. Department of Health, Education, and Welfare.*

Probability and statistics in the twelfth year?*

GEORGE GROSSMAN, *Chairman of Mathematics*
William Howard Taft High School,
New York, New York.

For the mathematically talented, this writer prefers other courses to the one in probability and statistics.

I HAVE READ the report of the Commission on Mathematics with interest, and I hope that within the next ten years most of its recommendations will be standard practice. But as to their experimental textbook[1] on probability and statistical inference and their recommendation that introductory probability with statistical applications become a second-semester twelfth-year alternative course of study, I am not so sure, at least with respect to the emphasis on statistical inference, that I would wish to emphasize so much this branch of applied mathematics at this time.

Let me first state that I do consider this experimental textbook prepared by the Commission to be a service to mathematics education. Since much of its contents is unfamiliar to most secondary-school teachers of mathematics, this booklet gives a masterly presentation of the Commission's idea of what should be taught in this second-semester course as an alternative course to an introduction to modern algebra.

The Commission believes it desirable that materials of instruction on statistics be introduced into secondary-school curricula. Here I agree completely. With our citizens continuously being bombarded by all kinds of statistical statements and arguments, verbal and graphic, we owe it to our youth to prepare them for able analysis. I would like to see, from Grade 7 onward, selected units involving statistical concepts taught in social studies, in mathematics, and in English. An introduction to statistical thinking *is* an important supplement to inductive and deductive thinking.

My first thesis is, then, "Statistics as a subject for general education *is* important."

The contents of the experimental textbook under discussion are important, too, and somewhere during the education of our mathematically gifted student, this content too should be emphasized. The questions are: where? how much? when?

My second thesis is "For the mathematically gifted student, the twelfth year may not be the most efficient place to take time from other topics."

Dr. Conant defines as the "academically talented" that fraction of the student body in our high schools who are able to study effectively and rewardingly a wide program of advanced mathematics, science, and foreign language. This group comprises the upper 15 per cent of our high school population, and includes as a subset the upper 3 per cent usually referred to as the "academically gifted." For this 15 per cent, Dr. Conant recommends,

* A speech delivered during the April, 1960, NCTM meeting in Buffalo, New York.

[1] *Introductory Probability and Statistical Inference, an Experimental Course Proposed for the Commission on Mathematics* (1959).

among other subjects, four years of mathematics, and for those in this "academically talented" group whose major field of interest is *not* mathematics, there could be incorporated in the twelfth year a rich unit in statistics to enable these students to use statistical reasoning in their future college courses and in research.

My third thesis is "For the academically talented student with major interests outside of the field of mathematics, taking four years of high school mathematics offers an opportunity to include units on statistics in the twelfth year—and such courses should be offered."

I am, however, more concerned with a subset of the academically talented student body of our secondary schools. This group of students includes the mathematically talented, the students who early in their high school career show an ability for abstract mathematics. This group contains the students who in college will major in mathematics, engineering, the physical sciences, statistics, operations research, computers, automation, or any field where the more mathematics learned in high school and in college, the better. For this group of students, I do not think that a course in probability and statistical inference has as much to offer as three other courses I shall describe.

It is for these students that I take what might be considered "an opposing view."

You will notice that in these preliminary remarks I have separated "probability" from one of its applications, "statistical inference." Chapters 4 and 5 of the experimental textbook are in my opinion of higher priority than the sections devoted to statistics. If we think it important to give these students a good foundation in the algebra of sets among other topics, then, as an application of set theory, the newer definition of "the probability of an event" in terms of "sample spaces" *is* certainly valuable. And if we consider the study of "abstract mathematical systems," such as groups and fields, important, then the axiomatic treatment of probability, first developed in the 1930's by Kolmogorov, is a desirable unit to introduce, and it is useful to develop from a few postulates some of the major theorems of probability theory. The contents of chapters 4 and 5 can therefore be said to belong in the twelfth year.

My seeming opposition to the other chapters in the Commission's experimental textbook and their contents is only one of priority. There is so much that we would like to offer the mathematically gifted student in years 9 to 12 that by including any one topic we imply the exclusion of some other topic. The inclusion of this course in year 12, therefore, means that we must place less emphasis on some other course or topics.

What are some of the topics that I feel the mathematically talented students can better be exposed to with the exclusion of statistical inference?

ALTERNATIVE I

For the upper 3 per cent—and my statistics are inaccurate here—of our "mathematically talented" students who show promise of becoming our future Ph.D.'s in mathematics, in the physical sciences, or in some other field where advanced mathematics is a tool subject, the sooner they begin their advanced mathematics courses in college the better. For these students we might consider analytic geometry and calculus as high school subjects. These students should take the advanced placement examinations during year 12 so that in their freshman college year they can use the calculus in their mathematics and science courses.

At my present stage of thinking, I would prefer that the Mathematics Advanced Placement program be limited to the potential college and graduate school major in mathematics, engineering, or the physical sciences, with sufficient physical, emotional, and social power and maturity. There is much to be said for not being in such a hurry to rush into advanced calculus in year 13. Advanced mathematics

courses require more maturity, and the extra year carries with it greater long-range understanding and appreciation.

Alternative II

For the majority of the mathematically talented student body, I would prefer a course in year 12 that would prepare these students for their college and graduate courses in mathematics. A course that would concentrate on fundamental concepts and foundations of mathematics, a course that would include and expand much of the second alternative of the Commission's *Introduction to Modern Algebra* is what I consider of higher priority. And if the course I shall describe meets with favor, I would recommend that an experimental textbook, similar in objective to the Commission's first alternative text, should be prepared to assist teachers of mathematics. (The SMSG's new *Introduction to Matrix Algebra* which I have just seen does aim in this direction.)

Two major objectives of this course are: (1) abstract mathematical systems and (2) number systems.

I have been doing experimental teaching in this direction. In the NSF Summer Institute which I gave at Columbia University in 1959 (and which I have attempted to describe),[2] and in the Science Honors Program of Columbia University's School of Engineering, I am giving such a course on Saturday mornings during the school year to selected high school students. Also for the past few years at the William Howard Taft High School in New York, I have experimented with units of this course.

As those among you who took the IBM tours in New York City during the NCTM's Christmas meeting in New York know, I have been experimenting with a course in "Programming for Automatic Digital Computers," using the IBM 650 at Columbia's Watson Laboratory. There we have been provided with machines, machine time on the 650, teaching literature and manuals, and IBM cards by the millions. I can see as an Alternative III to my Alternatives I and II, a course to be taken preferably *in addition* to Alternative I *or* II, a course in programming for automatic digital computers, with applications from various fields of mathematics including the theory of numbers. Here is where I would include a unit on statistical inference, for here we can combine statistics with programming, using the computer to perform the arithmetic calculations needed in statistics. Here is where I think statistical inference belongs in the education of the mathematically talented—in a course in applied mathematics to be taken in addition to, not instead of a course in pure mathematics.

Whenever I think of pure vs. applied mathematics, I am reminded of the quotation of David Hilbert in Jacques Barzun's *The House of Intellect*,

> We are often told that pure and applied mathematics are hostile to each other. Pure and applied mathematics are *not* hostile to each other. Pure and applied mathematics have never been hostile to each other. Pure and applied mathematics will never be hostile to each other, because in fact there is absolutely nothing in common between them.[3]

My Alternative II in pure mathematics is *my* first choice for the academically and mathematically talented students during the second semester of year 12.

I shall outline chronologically my order of topics:

Beginning with the question, "How can an automatic digital computer add two numbers electrically?" I develop the algebra of switching circuits. I have found this a wonderful introduction to "algebra" and "algebras." We discuss the use of variables "A, B, C" to represent switches, these variables being assigned the values

[2] "A Report of a National Science Foundation Summer Institute in Mathematics for High School Students at Columbia," The Mathematics Teacher, LIV (February, 1961), 75–81.

[3] Jacques Barzun, *The House of Intellect* (New York: Harper, 1959), p. 170.

"0" or "1," "0" representing the switch in one of its two positions, usually "open," and "1" the other position, usually "closed." Two switches in "parallel" and two switches in "series" give us our binary operations which we denote by "+" and "×." We discuss our right to call these "operations" and to denote them by the symbols "+" and "×." We discover inductively and then prove properties of "+" and "×" from the following postulates:

(1) $A=0$ or else $A=1$, for all A

(2) $0+0=0$; $0+1=1+0=1+1=1$

(3) $0\times0=0\times1=1\times0=0$; $1\times1=1$.

Various properties of "+" and "×," which look symbolically like the commutative and associative laws for the addition and multiplication of numbers in our more familiar algebra, are proved by complete induction (similar to "truth tables") since only four or eight cases are involved. We discover two distributive laws, the "idempotent" laws: $a+a=a$ and $a\times a=a$; we discuss the conclusion that the only coefficients and exponents we need are 0 and 1, and point up the two different meanings for "0" and "1" gained so far. We discover a complete duality between "+" and "×"; we discuss the use of "=," and equivalence relations, and their three properties become part of all future discussions.

We next apply the algebra of switching circuits to the design of simple circuits, first the two-way switch used by homeowners in their "upstairs-downstairs" switches, then we extend this to a three-way switch. We then turn to our motivating problem and develop the half-adder and full-adder in the binary scale of notation, which we take up briefly. (It is interesting that the sum in the half-adder leads to the same circuit as the two-way switch developed earlier, and the sum in the full-adder has the same circuit as the three-way switch.)

In Alternative III we could go more deeply into the algebra of switching circuits where as an optional topic, the design and analysis of more complicated circuits to solve more complicated problems, can be an end in itself. Individual students can pursue this for the designing of circuits to do additions and subtractions with signed numbers, for example.

But for us, our objective has been realized. We have been talking about A, B, C, $+$, X, $=$, calling this an "algebra," and we are ready for other algebras. By studying other algebras we can better appreciate the one we have been living with for so many years.

We are ready for abstract Boolean algebra. I like to begin with Huntington's postulates, in a way similar to the development in Eves and Newsom, *The Foundations and Fundamental Concepts of Mathematics.*[4]

In developing this, I include the associative laws for "+" and "×" as postulates, as did Huntington at first. After proving a few theorems, using some of his other postulates, we *prove:* $A+(B+C)=(A+B)+C$!

At this point we stop to discuss the requirements of postulational systems, the necessity for consistency, and the aesthetic desirability for independence. As a reading project, I assign the history of the parallel postulate, recommending the Dover reprint of Heath's *Euclid's Elements.*[5]

Parenthetically, throughout such a course (as in all advanced mathematics courses) a major objective *should be* reading! I encourage my students to explore the Dover and other catalogs, to explore the many books in our departmental libraries (courtesy of the NDEA, Title III), and to brouse in mathematics libraries. *The World of Mathematics* and the *Encyclopaedia Britannica* are wonderful sources of mathematics articles, and can be referred to wherever pertinent to develop the habit.

[4] Rinehart, 1958.

[5] Heath, T. L., *The Thirteen Books of Euclid's Elements.* Dover reprint 1956, Vol. 1, pp. 202–220.

After proving about ten theorems in abstract Boolean algebra where the elements, operations, and equivalence relation are undefined, I distribute a mimeographed sheet with fifteen examples of models or representations of Boolean algebras. The preliminary edition of *Universal Mathematics*, Part II, on "Structure in Sets," distributed a few years ago by the M.A.A., is a good source. We then discover that we have, if each of these fifteen models is a Boolean algebra, just proved 150 theorems! Nothing can better illustrate to these students the power of abstract mathematical systems than this!

We first return to our algebra of switching circuits to verify informally that we do have here a Boolean algebra (by showing that all of Huntington's postulates are valid here). The advantage in having proved the associative law shows up because it does not have to be verified separately.

We turn next to symbolic logic, first as a Boolean algebra, using disjunction and conjunction as our $+$ and $\times$, and note the parallelism between: (1) the algebra of switching circuits and its binary operations of parallel and series circuits, and (2) the algebra of symbolic logic with its binary operations. Switches taking only one of the two defined values of 1 or else 0, and propositions having only one of the truth values T or else F, point up the *isomorphism* between these two systems. The concept of isomorphism is carefully treated at this point and wherever it shows up later in the course.

This seems to be a good opportunity to present a brief unit in symbolic logic, in a manner similar to the treatment in texts by Allendoerfer and Oakley and by Kemeny *et al.* We spend time on *modus ponens* and *modus tollens*, on converses, inverses, and contrapositives of conditionals, on the law of the contrapositive and the law of the excluded middle. We practice proving "tautologies" using truth tables as our postulates. I like to introduce the Sheffer Stroke truth functions, to explore all sixteen possible truth functions of two variables, and to show how all of the others can be defined first from two undefined—taking Whitehead and Russell's choice of disjunction and negation, then Rosser's[6] choice of conjunction and negation, and then Church's[7] choice of conditional and negation. Then by using either of the Sheffer Strokes alone, we define all the rest. A taste of *Principia Mathematica* can be introduced to show how Russell and Whitehead developed *their* algebra of symbolic logic from undefined terms, relations, the operations of disjunction and negation, and five postulates. I suggest the exploration of other axiomatic treatments using Rosser's and Church's developments. Eves and Newsom's text has an excellent unit on this.

These explorations serve to introduce the student to important concepts of modern mathematics and can plant seeds that will grow during their college and graduate school careers.

The algebra of sets serves as our third model of a Boolean algebra, this one not isomorphic to the first two. Much of the algebra of sets can be assigned as self-study by this time, and the definitions interweaving the symbolism of the algebra of sets and the algebra of symbolic logic are worth the time and emphasis spent on this unit.

Two other models which are interesting because of the isomorphism between them are:

Model 4: the set of divisors of a number like 6 which is the product of exactly two different prime factors, with "$+$" and "$\times$" defined as the l.c.m. and g.c.d. of the two numbers of the set. By setting up the two "multiplication tables" for "$+$" and "$\times$" we can verify a few of Huntington's postulates, assigning them to students as exercises, and again I point out that we do not need to verify either associative law.

[6] J. B. Rosser, *Logic For Mathematicians* (New York: McGraw-Hill, 1953).

[7] A. Church, *Introduction to Mathematical Logic* (Princeton, N. J.: Princeton U. Press, 1956).

Model 5: the set of four ordered pairs with elements 0 and 1, with "$\oplus$", "$\otimes$", and "$\ominus$" defined by:

$(a, b)\ominus(c, d)$ if and only if $a=c$ and $b=d$ (expressed symbolically)
$(a, b)\oplus(c, d)\ominus(a+c-ac, b+d-bd)$
$(a, b)\otimes(c, d)\ominus(ac, bd)$

pointing up the two different uses of $+$, $\times$, and $=$. The use of circles around the defined "$+$," "$\times$," "$=$" helps here.

By writing on the blackboard *mod.* 5's multiplication tables, too, we discover the isomorphism between *mod.* 4 and *mod.* 5. I challenge my students to discover the explanation for this, and the student who does is thrilled by *his* discovery.

This completes the first half of my Alternative II, the first half being devoted to Boolean algebras, with special emphasis upon three models, each with a goal in mind: switching circuits to show how we can add numbers electrically (and if time permits, we design and build a calculator which can add any two numbers, no matter how big, as long as neither exceeds 3); symbolic logic and the algebra of sets being important goals in themselves and having been well motivated. But more important, we have been talking *about* mathematics—postulational systems, equivalence relations, binary operations, isomorphisms between different systems, and the power of abstract mathematical systems: what is true for the abstract system is true for each of its representations. We are learning mathematics and the nature of mathematics.

Part 2 of my Alternative II is a study of number systems, leading to a study of groups, rings, integral domains, and fields. Beginning with Peano's postulates for the set of natural numbers, we develop properties of natural numbers. We then define the set of integers as ordered pairs of natural numbers, with definitions given for "$=$", "$+$", "$\times$." Is our defined "$=$" an equivalence relation? Do our defined binary operations have the properties we would like to associate with the symbols "$+$" and "$\times$"? Are they well-defined, commutative, associative? Is "closure" a property? Do they have identities and inverses for these identity elements? The abstract mathematical system called an "integral domain" is defined and developed, and here, for the first time for many of these students, is the proof that "$(-a)\times(-b)=(+ab)$" developed.

We continue with definition of the set of rational numbers as ordered pairs of the newly defined integers, and the reason why we must exclude any such ordered pair whose second element is the integer "0" can be made apparent and necessary in order that the postulates of a *number field* (which we now define) can be satisfied. Ordered pairs of real numbers yield the set of complex numbers, and high school students can discover that there is nothing "imaginary" about the square root of minus one!

Generalizing from "ordered pairs" to "ordered triples," "ordered quadruples" and "ordered n-tuples," and to two-dimensional arrays, we generate hypercomplex numbers such as quaternions, Cayley numbers, and matrices. The newly developed SMSG textbook on matrix algebra gives teachers of mathematics the means whereby, either in class or as self-study, students can find an introduction to the theory of matrices and some of its applications. The abstract mathematical system, *linear vector spaces,* can round out this unit to give our students a clearer understanding of number systems, their definitions and properties, as well as about abstract mathematical systems.

One gap in this development becomes the core around which we can *prepare* our students *for* the calculus. The major role of the secondary schools is to prepare these students adequately for their future studies in college. For our potential teachers of mathematics, Ph.D.'s in mathematics, and future leaders in mathematical research, I have been describing two alternatives for the twelfth year, assum-

ing sufficient coverage in years 9, 10, and 11.

Either give them a thorough course in the calculus and analytical geometry, so that, by the Advanced Placement Program they can skip the freshman mathematics courses and begin with more advanced courses, or else prepare these students for their *college* calculus courses.

A study of the real number system, the hole in the above development, can help. The definition of "real numbers" using the Cantor development, where convergent infinite sequences of rational numbers define a "real number," and the set of all of these "regular sequences" yields the set of real numbers, is one way to motivate a careful treatment of the theory of limits. A careful treatment of the theory of limits is fundamental to the student's understanding of the differential and integral calculus, and is usually slurred over in freshmen calculus courses in college. If, by a careful development of the real number system we can at the same time motivate and develop the theory of limits, we shall be accomplishing our major objective of preparing these students for their college courses in the calculus.

Rather than to use year 12 to develop the statistical inference application of probability theory, I would prefer the mathematically gifted student be given insight into the nature of number systems, abstract mathematical systems, the theory of limits, and mathematical reasoning.

This to me seems to be of higher priority than the Commission's course in probability and statistical inference.

Letters to the editor

Dear Editor:

In the December, 1960, issue of THE MATHEMATICS TEACHER, pages 627–31, I introduced a method of testing the divisibility of numbers. All of the prime numbers below 100, except 83 and 97, appeared in the tables that I formulated at that time.

After reading the article, Mr. J. M. Elkin of Long Island University, Brooklyn, New York, suggested extending Table 1, to include $(x+25y)$ to test for 83 and Table 2, to $(x-29y)$ to test for 97. And in a letter in the May, 1961, issue of THE MATHEMATICS TEACHER, Mr. George S. Cunningham of Concord, New Hampshire, has indicated that using the distribution $(4x+y)$ when the integer is in the form of $(1000x+y)$ is a test for 83 and $(3x+y)$ when the integer has the form of $(100x+y)$ is a test for 97. Both of these suggestions are very good, I think.

By using a simple method of determinants, I have found a general equation that provides the distribution needed for any predetermined modulus. In this way, divisibility can be tested for every number.

When the integer has the form $(10x+y)$ then let $(ax+by)$ be any possible distribution. Since $10x+y\equiv 0 \pmod{m}$ and $ax+by\equiv 0 \pmod{m}$, the determinant

$$\begin{vmatrix} 10 & 1 \\ a & b \end{vmatrix} \equiv 0 \pmod{m}.$$

The expansion of this shows that $10b-a\equiv 0 \pmod{m}$. With such a relationship established, it is comparatively easy to ascertain the distribution required for any modulus.

For modulus 83, we determine that $a=7$ and $b=9$. Thus, the form $ax+by\equiv 0 \pmod{m}$ becomes $7x+9y\equiv 0 \pmod{83}$. Since this is an awkward arrangement for computation, we can subtract it from the original form $10x+y\equiv 0 \pmod{83}$ and use $3x-8y\equiv 0 \pmod{83}$. The actual testing for divisibility now follows the same algorithm that I developed in my article, with the one step of multiplying "x" by 3 before subtracting the multiple of "y" being added.

In a similar manner, the arrangement $3x+10y\equiv 0 \pmod{97}$ is shown to be a means of testing divisibility by 97.

The method of proving these distributions was shown in the article published in December. At that time, it was shown to be flexible enough to accommodate any distribution. It is also to be noted that this determinant method may be used when the integer is in the form $(100x+y)$ and $(1000x+y)$, by making the appropriate change in the determinant.

With the additional distributions now presented, a number can be tested for its divisibility by every prime number below 100. Many of the primes have several distributions possible, these latter being the result of using this determinant procedure.

SISTER M. BARBARA STASTNY, O.S.F.
Mount St. Francis, Box 1059
Colorado Springs, Colorado

Suggestions to the applicant for a National Science Foundation institute

W. H. MYERS, *San Jose State College, San Jose, California.*
—from a mathematics teacher serving in a department that has conducted at least eight such institutes.

HAVING BEEN CHARGED with the selection of participants for two National Science Foundation Summer Institutes and having served in a department that has conducted at least eight such institutes, the author has been impressed by the many things which applicants do or fail to do which result in their being denied the right to participate. Perhaps a listing of some of the common errors will aid teachers in making successful applications in the future.

These comments are meant to apply to all three types of institutes: summer, in-service, and academic year. However, by far the greatest bulk of applications processed are for summer institutes, and their processing unfortunately must be accomplished by the director in the most limited time. Admittedly, selection policy varies with the type of institute, the school, and the director. An omission which may be disqualifying at one institute may not so be at another.

Your application for an institute is important to you and justifies every care you can exercise. If successful, it can increase your knowledge of mathematics considerably, it can increase your value to your school and to your students, and it can increase your income by hundreds of dollars. Try to be accurate, be as neat as possible, and avoid such errors as misspelled words. Be certain to complete the application form in minute detail. Don't, for example, fail to list your birth date.

When you are asked to record the minimum one-way distance from your home to the institute, do just that. Directors may be inclined to discard an application if the distance shown is substantially more than the actual distance, not just because of this error, but because of the possible implications of this type of mistake.

Be careful in the listing of your dependents. If your spouse or any child earns more than $600 per year, you must not list this person for a dependency allowance. An applicant has shown the occupation of his wife as a secretary and then later tabulated her as a dependent. The director has difficulty resolving this problem.

List your experience carefully and as completely as called for. This means all work experience, not just teaching experience. An unexplained gap of a year or two may give rise to questions, such as these, in the mind of the director. Did the participant stop teaching for further study, for financial reasons, for health reasons, or because of a dislike for the profession? Why did he return to teaching? The answers to these questions may shed some light on the applicant's seriousness of purpose as a teacher.

In tabulating your education, be certain that your stated major is precise. If a

director finds an applicant who lists himself as a mathematics major on one page and then fails to list sufficient courses on another page to constitute a mathematics major, a suspicion of the responses is doubtless justified. Be certain to list under "College or University Education" all summer sessions and evening sessions which you have attended, whether subsidized or not. If you show no summer schools attended and then list mathematics courses taken after graduation, a reasonable question arises as to whether or not your courses were part of an earlier institute.

Carefully list all previous institute programs and identify them as summer institutes, academic year institutes, or in-service institutes. In showing your courses, make certain to place an asterisk before institute courses. The director may be giving especially favorable consideration to persons who have taken the trouble to participate in in-service training programs and yet look unfavorably upon the applications of those with either an academic year institute or two or more previous summer institutes. Incidentally, in the opinion of the author, failure to list a previous institute in which the applicant participated should be sufficient basis for refusal to pay the stipend, even after acceptance into the institute.

In listing the courses in mathematics which you have taken, list them *all* by both number and title, from the most elementary to the most advanced. If a course has been repeated, show the grade received each time. The director may be looking for persons who are at least through calculus. Don't hesitate to use a supplementary sheet in order to be meticulous about showing your record in mathematics courses. If you don't know in what year you took the courses, look up your record. If you are careless about even a relatively small item, such as this, the director may be justified in losing interest in your application. When you fail to tabulate the grades you received, the director may wonder whether your grades were so low that you hesitate to be frank. This can, in some institutes, cause disqualification.

The brief essay about yourself is a most important part of the application. Remember, the director and his staff will have the tremendous task of sorting out forty or fifty candidates from possibly a thousand or more applicants for a summer institute. After an initial sorting to eliminate those who simply do not meet the stated minimal requirements of the institute, a most careful study will be made of the applications. The contents of the essay may serve as the basis for selecting the final group from, say, two hundred otherwise eligible applicants.

At this point it is very important that you indicate clearly and concisely why participation in this particular institute is of vital importance to you and to the welfare of your students. Occasionally an applicant states that he hopes to get from a particular institute things which are in no way connected with the objectives of that institute as outlined in the brochure. For example, one man indicated that from an institute offering courses in algebra and geometry he expected to learn how to program for an electronic computer.

If you are working on curriculum reorganization and/or hold an appointment from which you can be especially effective in determining the direction which mathematics will take in your school, be certain to so state. The director is not interested in the fact, and should not be told, that you and your family would like to vacation in his area.

As a final note in this connection, don't append to your application a curriculum study which you have recently completed. Don't enclose a photograph of yourself unless specifically requested to do so. The author has known directors to throw out applications because of the possible implications of such an act. If the director has not asked for transcripts, don't send them.

If he asked for one letter of recommendation, send one and no more. If he has not asked for a letter of recommendation, don't feel that you are furthering your application by troubling him with additional mail.

Please make sure that you sign your application. During the period when selections are being made for a summer institute the director is busy beyond belief. There may not be time to return your application for signature. Above all, please don't attempt at this stage to engage the summer director in correspondence; he has all he can do to meet the deadlines. The selection of participants for an academic year institute or an in-service institute frequently does not have to be made in such a limited period, about twenty days between the last date of application and the date of notification for a summer institute. In connection with either of the former, there is often time for a necessary and desirable exchange of correspondence between director and potential participant. If you are acquainted with another member of the staff at the institution to which you are applying, before you write to this second individual to enlist his aid in insuring that your application receives special consideration, give some thought to how this action will be construed by the director.

Get your application into the hands of the director as early as possible. Some directors (not at the author's college), faced with the monumental task of selecting candidates for a summer institute, have been known to select the first fifty persons whose applications met the requirements of the institute. This is undoubtedly an exaggeration, but there is little question that the reading of the applications generally begins before the last of the applications are received in the mail. If an application for a summer institute is postmarked after the deadline date for filing, it may not even be read by the director.

Many directors enclose with the standard application form, in response to your inquiry about their institute, some type of local form, perhaps pertaining to housing during the institute. Failure to complete and return this supplementary form may disqualify your application if the time is short. Thus the only place to secure the forms by which to apply to a particular institute is from the director of that institute.

If the institute director is providing housing for the participants as a group, do your level best to take advantage of this opportunity to live with as well as to work with your fellow-teachers. Possibly as much as two-fifths of the value which you will receive from the institute will come from the experience of sharing burdens with other persons who face the same problems in their teaching as do you. Most directors, if some provision has been made for group housing, hesitate to select candidates who plan to commute daily to the institute.

You may choose to apply to several institutes. In this case, it is wise to make an original application to each of the several institutes. In some institutes the directors automatically exclude from consideration any application which is run off on a duplicating machine or which is a carbon copy. One applicant included in his essay the statement, "Your courses Mathematics 100 and 101 are just exactly what I need to round out my education," when in reality the institute to which he was applying wasn't offering courses carrying these numbers.

If in the institute brochure a stipulation is made that the institute is available to teachers of college mathematics and you are teaching in high school, then don't apply. The director cannot consider your application, however deserving it may be. If the institute is designed for teachers of mathematics in grades 10 through 12 and you are neither teaching nor supervising mathematics or are teaching mathematics only for ninth-graders, then don't bother to apply. Your application simply cannot,

under National Science Foundation regulations, be considered.

Once your application is complete in all detail and posted well before the deadline, just sit back and relax. It may be unwise to attempt to engage the director in individual correspondence at this, one of the busiest times in his life. He is as interested in completing the selection of his participants as you are in becoming one of them. He will notify you of your selection as a participant or as an alternate just as soon as regulations permit him to do so.

If you are so fortunate as to be granted a stipend, you are going to enjoy a tremendous experience. Come to the institute with the attitude that for the duration of the institute you are again a student. It is especially difficult for teachers as a group to visualize themselves reverting to the status of students, yet at the institute they are in truth students, students of intellectual and emotional maturity who will be treated with respect by the professors in the institute.

Your opportunity for self-advancement is being made possible by the contributions of the American taxpayer. You will be expected to work diligently on your courses to justify this expenditure of money on you. Please don't come to an institute with the thought that you will be allowed to audit the courses and enjoy a vacation while the institute is in session. For the funds you are receiving, you should be willing to put your chips on the table and enter the game. Do your very best, take the grades which you earn, and then try to realize the full extent of the wealth of knowledge and experience you have gained from your institute courses and associations.

Have you read?

BEARDSLEE, DAVID C., AND O'DOWD, DONALD D. "The College-Student Image of the Scientist," *Science*, March 31, 1961, pp. 997–1001.

It is always interesting to know what a person thinks of other people. This is why gossip is popular. High school students will like to know what college students think of scientists. This article gives them an answer.

To the college student, the scientist is highly intelligent, radical in social outlook, depressed, indifferent to people, hard to get to know, controls his impulses with difficulty, and is a little strange. How does he compare with other professions? He is a lot like college professors, somewhat like school teachers, and almost the opposite of personnel directors. In this article there is some information on how college faculties looked at scientists and how students choosing an occupation made that choice. For example, only 3 per cent of the girls wanted a scientist for a husband as compared to 20 per cent who wanted doctors and 20 per cent wanting lawyers.

Read this article and then write one, offering suggestions as to how this rather unfavorable image might be changed. After all, we don't want all scientists doomed to bachelorhood.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

WATERMAN, ALAN T., "Science in the Sixties," *American Scientist*, March, 1961, pp. 1–8.

This is the Procter Prize address to the annual RESA convention in December. Dr. Waterman is director of the National Science Foundation, and in this position he has had cause to look carefully into the future.

Radical changes are taking place, research is no longer associated purely with teaching and the advancement of knowledge. The research community has broadened to take in the discovery of potential economic value, the international political problems, and the world's social and physiological problems. The post-war period for the first time has brought together engineers, mathematicians, biologists, and other scientists for team research. This brings together the fields of science and social science and leads toward development and production planning.

The future is bound up closely with capability in science. This means for us collaboration and co-operation with all the peoples of the world in promoting the pursuit of science free from political and social international tensions. This article is crystal clear as to our survival route. Read it and see.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Teaching seventh-grade mathematics by television to homogeneously grouped below-average pupils

JAMES N. JACOBS, JOAN BOLLENBACHER, *and* MILDRED KEIFFER,
Cincinnati Public Schools, Cincinnati, Ohio.

SEVERAL EXPERIMENTS in the use of television in direct instruction in the Cincinnati public schools have indicated that a single telecourse generally does not meet the needs of all pupils. In those experiments in required courses where there is a wide range of pupil ability, the typical outcome has shown a significant interaction between pupil ability level and method of instruction. In one experiment, pupils of above-average ability showed superior achievement through televised instruction in sixth-grade science, while below-average ability pupils achieved significantly better through conventional instruction [3]. A telecourse in driver education also showed high-ability pupils achieve significantly better by television [6]. Another series of experiments revealed that television was more effective for average-ability pupils in seventh-grade mathematics, but in sixth-grade science it was significantly less effective than conventional instruction for above-average and below-average ability pupils [7].*

RELATED STUDIES

Interest in televised instruction in relation to ability level has been explored by several other investigators. Two of these investigators [1, 4] working with the armed services have shown that although both high- and low-ability groups learned as much by television as by conventional instruction, low-ability groups learned relatively more on the basis of percentage gain from pretest to post-test. Fritz [2] in experimenting with army personnel found no differential effects between method of instruction and ability level of students. Studies conducted at Miami University [5] confirm the findings of the latter study. Williams [8], on the other hand, found that higher-ability students tend to benefit more from televised instruction.

The research conducted within the Cincinnati public schools and the results obtained from other investigations point up the contradictory and inconsistent nature of the findings. The range of ability present in the aforementioned investigations is at least one probable factor responsible for the inconsistent results. Ordinarily high- and low-aptitude students are arbitrarily defined by a division of the range of ability available for experimentation. Unless a sufficiently wide range of ability is available at the outset, it is likely that a distinction without a difference will result. Such a situation applies particularly to studies carried out at the university level or in certain high school courses in which enrollment is rather select.

DISCUSSION OF THE PROBLEM

The fundamental problem, however, is to determine whether the observed differ-

* Numbers in brackets refer to references listed at the end of the article.

ences in effectiveness of televised instruction in ability level are intrinsic or extrinsic in nature. Certainly the inconsistencies of research findings lead one to suspect extrinsic factors as responsible for the differential effectiveness. Although many extrinsic factors may be identified as reasonably accounting for the observed interactions (e.g., age, maturity of students, type of criterion instruments used, caliber of both television and classroom instruction, type of course taught), the factor selected for study in this investigation is the level at which instruction is directed.

The suggested hypothesis is that television instruction per se is not intrinsically more effective for pupils of above-average, average, or below-average ability. Rather it is hypothesized that the particular ability level which seems to benefit most from television instruction is simply that ability level to which the television instruction is aimed. One obvious implication is that pupils must be grouped homogeneously and that television instruction must be adjusted to the competencies of that particular group. Under such conditions it is hypothesized that television instruction will be significantly more effective than conventional instruction as measured by a standardized achievement instrument. The present experiment was designed to test this hypothesis.

Purpose of the experiment

The purpose of this experiment was to determine the relative effectiveness of television instruction directed to homogeneously grouped below-average pupils in comparison with similar groups taught within the framework of the conventional classroom.

Design and methodology of the experiment

This experiment was conducted in seventh-grade mathematics. The telecourse was designed for pupils who were one and one-half to two and one-half years retarded in arithmetic achievement. The television teacher had had experience in teaching mathematics to pupils of approximately this ability. In addition, the television teacher had one year of television teaching experience in seventh-grade mathematics previous to this experiment.

It would have been ideal from the viewpoint of experimental design to incorporate classes of average and above-average achievement into the experiment to determine whether conventional instruction which presumably could be adjusted more readily to varying levels was more effective with these groups. The instructional and administrative deterrents to such a plan, which would involve assigning some pupils to classes designed below their level, obviously made it infeasible.

During the spring of 1959, pupils with arithmetic grade-equivalents between 4.5 and 5.5 were identified. The Stanford Intermediate Arithmetic Achievement Test, Form K, given routinely in the sixth grade, was used as the basis for selection. The schools selected to participate in this experiment were those in which there were sufficient numbers of these pupils to form four classes with approximately 30 pupils in each class. Three of the Cincinnati junior high schools met these qualifications. Each principal was given a list of all the pupils whose arithmetic grade-equivalents were between 4.5 and 5.5 and was instructed to constitute randomly four classes. Two of these four classes in each of the three schools were assigned randomly to receive television instruction and the remaining two to receive conventional instruction.* Finally, two seventh-grade mathematics teachers in each school were assigned one class to be taught by television and one class to be taught in the conventional manner. It should be

* Due to certain extenuating circumstances, one television class was unable to view telecasts regularly and was, therefore, dropped from the experiment. The control class taught by the same teacher also was dropped from the experiment.

noted that comparison will be made between classes taught within the same school and by the same teacher. The control of these factors is considered important since teacher and school differences are usually of sufficient magnitude to decrease the precision of the experiment.

The telecourse covered one full year of instruction, although the criterion instrument was administered in March. Telecasts were presented every Monday, Wednesday, and Friday for a period of 20 minutes. The total time involved in television instruction was 60 minutes per week which represents approximately 24 per cent of the total instructional time devoted to seventh-grade mathematics. Each telecast was followed by approximately 30 minutes of follow-up and discussion by the classroom teacher in addition to a full period of classroom instruction on Tuesday and Thursday.

In March, or seven months after the experiment began, the Metropolitan Intermediate Arithmetic Achievement Test, Form Bm, for grades 5 and 6, was administered to both the television and nontelevision classes in the experiment. The standard scores obtained from this test were used in the analysis of the data and represented the criteria in determining the effectiveness of the two methods of instruction. Both the Computation and Problem Solving and Concepts sections of the battery were administered and separate analyses were made. Since all classes were organized on the basis of homogeneous achievement of pupils previous to the experiment, it is likely that any interaction of methods and teachers which might occur is due to extrinsic factors.

Basically the simple randomized design involved, in effect, five repetitions of the same experiment. The data were treated in a two-way analysis of variance involving methods of instruction and classes taught by the various teachers. Since unequal numbers were present in each of the treatment classifications, an unweighted mean analysis of variance was used. Since variation among pupils in the same treatment group was considered the estimate of experimental error, the harmonic mean was used for comparison of mean squares.

Results of the Experiment

The mean standard scores computed for the television and nontelevision groups on the Computation and Problem Solving and Concepts subtests are shown in Tables 1 and 2 for each teacher. The unweighted mean analysis of variance of the Computation subtest scores showed a lack of significant interaction ($F=.67$, *df* 4 and 253) and a lack of significance of difference between the two methods averages ($F=3.00$, *df* 1 and 253).

TABLE 1

Standard scores on the Computation Subtest of the Metropolitan Intermediate Arithmetic Achievement Test, Form Bm, for television and nontelevision classes taught by five different teachers, Cincinnati public schools, 1959–60.

Teacher	Computation				
	N	TV averages	N	Non-TV averages	TV—Non-TV differences
1	26	245.65	24	242.88	+2.77
2	27	250.96	23	249.43	+1.53
3	23	248.43	31	244.68	+3.75
4	33	252.45	21	250.90	+1.55
5	25	245.32	30	247.67	−2.35

TABLE 2

Standard scores on the Problem Solving and Concepts Subtest of the Metropolitan Intermediate Arithmetic Achievement Test, Form Bm, for television and nontelevision classes taught by five different teachers, Cincinnati public schools, 1959–60.

Teacher	N	Problem solving and concepts TV averages	N	Non-TV averages	TV—non-TV differences	t-Ratio
1	26	252.46	23	250.17	+2.29	1.22
2	26	251.65	25	247.48	+4.17	2.34*
3	23	252.74	31	248.52	+4.22	2.40*
4	31	250.84	23	248.57	+2.27	1.34
5	26	247.69	27	250.93	−3.24	1.98

* Significant at the 5 per cent level.

Since the over-all test of the null hypothesis was accepted, individual tests for each replication cannot legitimately be made. All differences between methods of instruction are considered to be due to chance variation and, therefore, lacking in statistical significance.

In Table 2 a similar analysis of the Problem Solving and Concepts scores revealed a significant interaction ($F=2.95$, df 4 and 251) between methods and teachers. In view of the significance of interaction, an over-all test for methods difference is meaningless, and it becomes necessary to test for the significance of methods differences for each teacher's classes separately.

The results of each test of significance of difference are shown in the form of t-ratios presented in the last column of Table 2. It will be noted that of the five differences between the television and nontelevision classes, four comparisons show differences favoring television while one shows a difference favoring conventional instruction. Only two of these five differences are significant. These two significant differences, both favoring televised instruction, are shown in the classes taught by teachers 2 and 3. The comparison favoring conventional instruction between the classes taught by teacher 5 is not large enough to be significant at the 5 per cent level. The fact that a significant interaction was noted in the Problem Solving and Concepts subtest and not in the Computation subtest is not surprising. The emphasis in the mathematics telecourse was on application and problem solving rather than computation.

Discussion

It is apparent from these results that the hypothesis that televised instruction would be more effective than conventional instruction under conditions of homogeneous grouping and specialized instruction must be rejected. These results, however, cannot be interpreted to mean that homogeneous grouping is ineffective, since both differences which were statistically significant were in favor of the televised method. The remaining three differences showed neither the televised nor the conventional method to be superior. In comparison with last year's experiment, which showed above-average ability pupils achieving significantly more under conventional classroom conditions, this experiment may be considered successful, since no difference significantly favored conventional instruction.

The fact that the televised method did not show consistent superiority probably reflects the varying ability of classroom teachers to adapt the telecasts to the ongoing instructional program. The classroom teacher's instruction in the television classes cannot be overlooked, since approximately three-fourths of the total instructional time was given by the classroom teacher and only one-fourth devoted to televised instruction. Under such conditions, the power of televised instruction would have to be great in order to be reflected in significant differences in achievement test scores.

This discussion raises the interesting question of whether the power of television instruction, i.e., its ability to differentiate achievement level, is greater when used with homogeneously grouped above-average ability pupils than when used with homogeneously grouped below-average pupils. It is planned that this inquiry be investigated in future experiments with television instruction.

It is not known definitely what classroom teacher attributes are necessary to utilize television instruction effectively. It will be noted from Table 2 that the two largest differences favoring the television method were made by teachers 2 and 3, both teaching in the same school. Both of these teachers are known to be enthusiastically in favor of television instruction. In addition, the principal of this school has shown unusual interest and enthusiasm in televised instruction. It is difficult to determine the importance of these factors in the successful use of televised instruction, although it is reasonable to assume they are of some importance.

Conclusions

1 Televised and conventional instruction were equally effective in imparting computational skills in seventh-grade mathematics to pupils initially below norm in achievement and grouped homogeneously. With respect to achievement in problem solving and concepts, a significant interaction between methods and teachers occurred resulting in two significant differences favoring television and three nonsignificant differences.

2 In comparison to the results of previous experiments executed under conditions of heterogeneous grouping, it appears that television instruction is more effective when pupils are grouped homogeneously. The validity of this generalization probably depends upon the range of ability or variability of the population under consideration.

References

1 Boone, W. F., *Evaluation of the U.S. Naval Academy Educational Television as a Teaching Aid.* Annapolis: United States Naval Academy, No. 7010, October, 1954. (duplicated)

2 Fritz, M. F., *et al.*, *Survey of Television Utilization in Army Training.* Port Washington, L.I., New York: Special Devices Center, Human Engineering Report Spec Dev Cen 530-01-1, December, 1952.

3 Jacobs, James N. and Bollenbacher, Joan K., "An Experimental Study of the Effectiveness of Television Versus Classroom Instruction in Sixth Grade Science in the Cincinnati Public Schools, 1956–57." *Journal of Educational Research,* Vol. 52, No. 5: 184–89, January, 1959.

4 Kanner, J. H., Runyon, R. P., and Desiderato, O., *Television in Army Training: Evaluation of Television in Army Basic Training.* (Technical Report 14) Washington, D.C.: Human Resources Research Office, The George Washington University, November, 1954.

5 Macomber, Glenn F., and Siegel, Laurence, *Final Report of the Experimental Study in Instructional Procedures.* Oxford, Ohio: Miami University, January, 1960.

6 *Report of an Experiment in Teaching Biology and Driver Education by Television.* Cincinnati, Ohio: Cincinnati Public Schools, 1957–58. (duplicated)

7 *Report of Three Experiments in the Use of Television in Instruction.* Cincinnati, Ohio: Cincinnati Public Schools, 1958–59. (duplicated)

8 Williams, D. C., "Mass Media and Learning—An Experiment," *Explorations,* No. 3: 75–82, 1954.

A solution for certain types of partitioning problems

M. H. GREENBLATT, *RCA-Astro-Electronics Division, Princeton, New Jersey.*

How can one use a 7-quart jug and an 11-quart jug to measure 2 quarts?

PROBLEMS INVOLVING PEOPLE caught with several containers of improbable size, and being further required to produce an unusual fraction of the contents of one of the containers, are not uncommon in textbooks. In one example, a maid is sent to a stream to get exactly 2 quarts of water but has only a 7-quart jug and an 11-quart jug available to her. Another involves two thieves who wish to divide their 8-gallon haul of whiskey evenly, but have only a 3-gallon can and a 5-gallon can available to them. Problems of this sort can be laid to rest by means of the following demonstration.

The Euclidean Algorithm[1] yields integers r and s which satisfy the equation

$$r\cdot p-s\cdot q=1, \qquad (1)$$

where p and q are relatively prime integers. This equation can be interpreted as saying that if a p-unit container is filled exactly r times and is emptied only into a q-unit container, and this q-unit container is emptied exactly s times (into the master container), then exactly one unit must remain in the p-unit container.

If the desired amount were not a single unit, but a units of the substance, then each term in equation (1) could be multiplied by a giving

$$a\cdot r\cdot p-a\cdot s\cdot q=a.$$

[1] Richard Courant and Herbert Robbins, *What Is Mathematics?* (New York: Oxford University Press, 1941), p. 42ff.

This equation indicates a way of obtaining a units, but it might not indicate the procedure requiring the minimum number of transfer operations. In such a case, there are two things which can be done. If, in the equation above, $a\cdot r>q$ and $a\cdot s>p$, then some multiple of $p\cdot q$ can and should be subtracted from each term on the left. This altered equation will then indicate a more economical solution of the problem. Also, the equation

$$r'\cdot p-s'\cdot q=-1 \qquad (2)$$

which can be obtained by judicious manipulation of (1), should be investigated. In this case, in which r' and s' are different from r and s, the roles of the p and q containers are reversed; i.e., the q container is filled s' times and the p container is emptied r' times. Again, in this case, the terms on the left in equation (2) should be decreased by a multiple of $p\cdot q$ if possible. Either this solution or the one derived from equation (1) will be the most economical solution. Equation (1) and equation (2) may lead to the same solution and then both are equally economical.

The question of whether the source is infinite (as in the case of the stream) or finite (as in the case of the 8-gallon cask) is not important because the size of the source does not enter into the equation. The only requirement is that the source capacity be at least $(p+q-1)$. If the size of the source is less than this, certain

mathematical solutions are not physically realizable. If the numbers p and q are not relatively prime, equation (1) is no longer true, certain numbers can then not be formed by multiplication by a, and the problem loses some of its interest.

The problem of obtaining 2 quarts of water, having only a 7-quart jug and an 11-quart jug to measure with, illustrates the method of solution. From the Euclidean Algorithm,

$$2\cdot 11-3\cdot 7=1.$$

As shown above, this can be changed to

$$(4\cdot 11)-(6\cdot 7)=2$$

and this equation states that at least $4+6$ filling and emptying operations are required to get two quarts if one starts by filling the 11-quart jug. To investigate the alternate solution, we proceed as follows:

$$\begin{array}{ll} & (2\cdot 11)-(3\cdot 7)=1 \\ \text{Multiply by 6:} & (12\cdot 11)-(18\cdot 7)=6 \\ \text{Subtract 7:} & (12\cdot 11)-(19\cdot 7)=6-7 \\ & \qquad\qquad\qquad\quad =-1 \\ \text{Subtract } 7\times 11\text{:} & (5\cdot 11)-(8\cdot 7)=-1 \\ \text{Multiply by 2:} & (10\cdot 11)-(16\cdot 7)=-2 \\ \text{Subtract } 7\times 11\text{:} & (3\cdot 11)-(5\cdot 7)=-2 \end{array}$$

According to this solution, $3+5$ filling and emptying operations are required to get two quarts if one starts by filling the 7-quart jug. This second solution is hence the more economical one for this problem. The total number of transfer operations is twice the number of filling and emptying operations minus 1. This is so because each filling operation must be followed by a transfer to the other container and each emptying operation, except the last, must be followed by a transfer from the other container.

Have you read?

HORTON, ROBERT E., "Learning Theories and the Mathematics Curriculum," *Mathematics Magazine*, November–December, 1959, pp. 79–98.

We can never learn too much about learning, and this article offers a few suggestions. There are conditioning, connectionism, field theories, and reorganization as theories of learning. The result of this effort is concept formation. Concepts are formed by generalizations; these are best formed with the aid of symbols so that the concept can be expanded as experience expands. Important phases of concept formation are establishment of hypotheses, organization, grouping, testing, accepting, or rejecting. Teaching so that the student forms such concepts includes also order, participation, memory in relation to mental processes, and the complex structure of problem solving. This is a valuable article for all of us to read and ponder. Are we developing concepts through a curriculum built on a psychological basis?—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

NORTON, MONTE S., "Some Needed Breakthroughs in Mathematics Education," *School Science and Mathematics*, October, 1960, pp. 533–538.

The author of this article recognizes that breakthroughs are occurring every day, but he wants to encourage a more rapid and refined process. He believes the areas of concentrated effort should include instructional materials where a dynamic classroom laboratory is developed. This can be done through flexible classroom space, listening stations, libraries of tapes, films and individual viewing facilities, and others. He believes more emphasis needs to be placed on self-evaluation, independent work, production of models, opportunity where needed, and library facilities and time.

The curriculum will be pointed to the challenge of mathematics itself, the nature of the system, and the nature of proof.

The dynamic classroom requires a variety of competences which can be met by team teaching, opportunities for professional growth, and better supervision. The author closes his article by noting that the public must be kept informed if progress is to be made. This article has lots of food for thought.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

A bibliography for careers in mathematics*

NURA D. TURNER, *State University of New York, College of Education, Albany, New York.*

Sources of information concerning careers in mathematics.

As a result of my administering the Annual High School Mathematics Contest sponsored by the Mathematical Association of America and the Society of Actuaries, in the upstate section of New York, and my contact with the students in that section who ranked high in the 1958, 1959, and 1960 contests, I have become aware that capable high school students interested in mathematics do not know what careers are open to them in that field. I am, therefore, listing some recent references that may be of help in acquainting both high school students and their teachers with careers in mathematics.

General information

1 Nourse, Alan Edward. *So You Want to Be a Scientist.* New York: Harper, 1960. $3.60. This book is planned for young boys and girls. It is simply but effectively written and can be read in an hour or two. The first half provides general information; the second half is devoted to a discussion, though not detailed, with respect to training and opportunities in the fields of mathematics and the physical, biological, and earth sciences.

2 Conference Board of the Mathematical Sciences, "Careers in Mathematics." The Mathematics Teacher, LIII (May, 1960), 340–43. In this article, teaching and industrial careers in mathematics are discussed with respect to satisfactions, requirements, nature of the work, and salaries.

3 Waterman, A. T. "Scientific Woman Power—A Neglected Resource," *Science Education*, XLIV (April, 1960), pp. 207–13. This is a general discussion of careers for women that would be well worth reading by high school girls interested in mathematics before they turn to reading about careers in detail.

4 "A Science Career for You," *National Business Woman*, XXXIX (May, 1960), pp. 6–7. This article provides a general discussion on opportunities for women in chemistry, physics, geology, astronomy, and meteorology.

5 "What Jobs in Science," *The American Biology Teacher*, XXII (March, 1960), p. 164. This article provides a look forward over the next ten years with respect to the need for both professionally and technically trained personnel.

Specific information

1 Angel, Juvenal L. *Careers for Majors in Mathematics.* New York: World Trade Academy Press, Inc., 1959. Address: 50 East 42nd Street, New York 17, New York, $1.25. This is a monograph of 30 pages which presents (1) the field of mathematics; (2) historical background; (3) personal qualifications; (4) nature of the work; (5) opportunities in mathematics; (6) getting started; (7) where employment is found; (8) remuneration; (9) women in mathematics; (10) educational background;

* This article was also published in *Science Education*, XLV (October, 1961), No. 4, 293–95.

(11) major fields of occupational specialization that include discussions of elementary, secondary, and college teachers, the industrial mathematician, the mathematical physicist, the mathematical statistician, the actuarian mathematician; the government mathematician; (12) advantages and disadvantages; (13) scholarships. Twenty-four references to occupational literature in the field of mathematics published from 1951 to 1959 are listed.

2 *Careers in Mathematics*, Research Number 262. Chicago: The Institute for Research, 1959. Address: 537 South Dearborn Street, Chicago 5, Illinois. Estimated cost, $1.00. This is a monograph of 24 pages which discusses (1) importance of mathematics; (2) where mathematicians are employed; (3) history of mathematics; (4) pure and applied mathematics; (5) shortage of mathematical students; (6) types of mathematical work and opportunities for employment that include teaching, mathematics in the armed services, exploration of outer space; (7) attractive and unattractive features of mathematical work; (8) personal qualifications; (9) qualifications for federal employment; (10) need of knowledge of modern languages and mathematics; (11) education; (12) scholarships; (13) opportunities for employment; (14) women in mathematics.

3 *Jobs in Mathematics*. Chicago: Science Research Associates, Inc., 1959. Address: 259 East Erie Street, Chicago 11, Illinois, $1.00. This 31-page booklet discusses (1) looking ahead to your career; (2) some careers in mathematics, including the actuarial field, work with electronic computers, teaching, and statistics; (3) what it takes; (4) how to qualify; (5) getting started; and (6) the future. Fifteen references for further reading are listed that were published from 1953 to 1959.

4 "Careers With Mathematics," *Jewish Life*, XXVIII (June, 1960), pp. 34–38. This is a quickly read article in which the careers of the statistician and actuary are discussed. Other topics include the use of mathematics in physics, fields suffering from a shortage of mathematicians, and the electrical computer as an example of applied mathematics.

5 *Occupational Outlook Handbook, Career Information for Use In Guidance*. The United States Department of Labor, Bureau of Labor, Statistics Bulletin, No. 1255, 1959. $4.25. This is a handbook of 785 pages. In it the careers of the mathematician, statistician, programmer, actuary, teacher, engineer, chemist, physicist, etc., are discussed with respect to (1) where employed; (2) training, other qualifications, and advancement; (3) employment outlook; (4) earnings and working conditions. One need not purchase the complete handbook. The material on mathematicians, statisticians, and programmers is available for 10¢ in Bulletin No. 1255-46; material on actuaries for 15¢ in Bulletin No. 1255-39; material on teachers for 15¢ in Bulletin No. 1255-83; material on engineers for 15¢ in Bulletin No. 1255-28; material on chemists for 5¢ in Bulletin No. 1255-16; and material on physicists for 5¢ in Bulletin No. 1255-61. All are available from Superintendent of Documents, Washington 25, D.C.

6 The Mathematical Association of America. *Professional Opportunities in Mathematics* (5th ed.). Buffalo, N.Y.: University of Buffalo, 1961. This is a booklet of 32 pages which discusses (1) the teacher of mathematics; (2) opportunities in mathematical and applied statistics; (3) the mathematician in industry; (4) mathematicians in government; (5) opportunities in the actuarial profession. Listed are organizations that employed ten or more persons listed in the 1960–1961 combined membership of the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics; 25 references for

further reading. Copies may be obtained from the Mathematical Association of America, University of Buffalo, Buffalo 14, New York, at a cost of 25¢ for a single copy or 20¢ per copy for an order of five or more.

7 Turner, Nura D. "A Day with Mathematics As a Career," *New York State Education,* XLVIII (October, 1960), p. 25. This article describes the program arranged for the 31 students who ranked in the top one per cent in the upstate section of New York in the 1960 Annual High School Mathematics Contest sponsored by the Mathematical Association of America and the Society of Actuaries. The program was planned to show these outstanding students how they could make a career in the expanding field of mathematics.

8 The National Council of Teachers of Mathematics and The National Academy of Sciences-National Research Council. *Careers in Mathematics.* Washington, D.C., 1961. Address: 1201 Sixteenth Street, N.W., $0.25. In this booklet of 28 pages there are short biographies of eight young mathematicians, followed by: (1) a short discussion of careers involving the training of only high-school or technical institute mathematics; (2) a more detailed discussion of the careers of engineering, analysis, programming, and coding, pure and applied statistics, teaching, research in industrial and pure mathematics, and government service. Listed are the names of 12 mathematical organizations and 39 mathematicians, sources for obtaining further information.

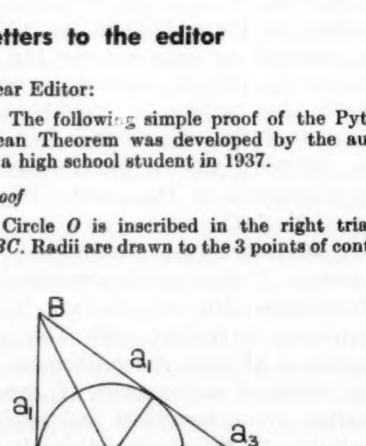
Letters to the editor

Dear Editor:

The followi g simple proof of the Pythagorean Theorem was developed by the author as a high school student in 1937.

Proof

1. Circle O is inscribed in the right triangle ABC. Radii are drawn to the 3 points of contact.



$$\begin{cases} a_1+a_3=c \\ a_1+a_2=a \\ a^2+a_3=b \end{cases}$$

2. The sum of the areas of all the parts of ΔABC is

$$a_2^2+a_2a_3+a_1a_2.$$

3. Since this is equal to the area of ΔABC, we have

$$a_2^2+a_2a_3+a_1a_2=\frac{ab}{2}=\frac{(a_1+a_2)(a_2+a_3)}{2}.$$

4. From this, we get

$$a_2^2+a_2a_3+a_1a_2=a_1a_3$$

and

$$2a_2^2+2a_2a_3+2a_1a_2=2a_1a_3.$$

Then,

$$(a_1^2+2a_1a_2+a_2^2)+(a_2^2+2a_2a_3+a_3^2)$$
$$=a_1^2+2a_1a_3+a_3^2$$

and

$$(a_1+a_2)^2+(a_2+a_3)^2=(a_1+a_3)^2$$

or

$$a^2+b^2=c^2.$$

Q.E.D.

Sincerely,

Irving Stein
Senior Physicist
Research Division, Ampex Corporation
Redwood City, California

• EXPERIMENTAL PROGRAMS

Edited by Eugene D. Nichols, Florida State University, Tallahassee, Florida

A study of pupil age and achievement in eighth-grade algebra

by Dorothy L. Messler, Dade County Schools, Miami, Florida

More and more, educators are evaluating curriculum programs to ascertain what can be done to meet the academic needs of pupils whose differences vary according to abilities and interests. One of the greatest problems has been in the field of mathematics, because it is one of the fastest growing and the most rapidly changing of the sciences. Many changes are taking place in the various school systems throughout the country to enrich and extend the mathematical program for the benefit of exceptional students.

Many teachers and parents are aware that the standard eighth-grade mathematics offers little challenge to the academically talented youngster. Some schools offer an enrichment program which most of the students feel is just "extra" homework and frequently call "busy work." These students become bored, and what is more tragic, some lose their zest for learning and develop poor study habits.

A number of teachers and parents agree with Dr. Conant's observations that the pupil who is above average in quantitative ability should take algebra in the eighth grade. On the other hand, there are some people who do not believe that eighth-grade students have the readiness or the maturity to understand algebraic concepts as well as a ninth-grader does.

In an attempt to secure evidence that would provide a definite answer to this problem, the Dade County school system planned an experiment at Palmetto Junior-Senior High School during the school year 1959–60. The planning team included the director of curriculum and instructional services, the supervisor of secondary mathematics, and the teacher who was to be involved in the experiment. With the co-operation of the local school administration, an analysis of pupil-data was initiated by the guidance department.

To determine the possible effects of chronological age and maturity on achievement in one year of algebra, two groups were selected for a year's study. One group consisted of 34 eighth-grade students, 21 boys and 13 girls. The other group consisted of ninth-grade students with 21 boys and 13 girls to match the eighth-grade group. The mean average of the younger group was 13.1 years, while 14.2 years was the mean of the older group at the beginning of their class activities in elementary algebra.

The students were carefully selected on the basis of I.Q., arithmetic achievement, academic ability in all subjects, and teacher and counselor opinions based on such factors as emotional stability, interests, work habits, and regularity of attendance.

The eighth-grade students were screened on the basis of the Otis Quick-Scoring Mental Ability Test, Beta, and the California Arithmetic Test, Intermediate, which had been given in the seventh grade. The group selected had a mean score on the Otis Test of 122.8, with a range from 110 to 136. The mean

percentile on the California Arithmetic Test was 92.3, with a range from the 85th percentile to the 99th percentile.

These same two tests were used in the selection of the ninth-grade group—the Otis Test had been administered to these pupils when they were seventh-graders and the California Arithmetic Test while they were in the eighth grade. The students were matched with the eighth-graders—pupil for pupil—so that the I.Q.'s of each pair differed by no more than five points. For the older group, the mean score on the Otis Test was 121.3 and the mean percentile on the California Arithmetic Test, 92.6 percentile. The range on each of these tests was the same as that of the younger group.

Selection for these classes began during the latter part of the school year 1958–59. As eighth-grade students had not previously been permitted to enroll in algebra, it was felt that the approval of the students and their parents should be obtained. The students' immediate, enthusiastic response in favor of studying algebra was encouraging. The parents were invited to the school for a discussion of the program. They were given an explanation of the selection of the students for this class and the purposes and plans for the future of these children in the area of mathematics. The parents were requested to give approval for their child's enrollment in the course. In each case, the child and the parent welcomed the opportunity to participate in the planned program.

As the ninth-grade students had already selected algebra as their mathematics course, the program was not discussed with these students or their parents at this time. However, the students had been in the class only a few weeks when they realized that it was a homogeneous group and that they were there by selection rather than by chance. These ninth-graders were enthusiastic about being in a class where, as they expressed it, they "were not bored by the slow pupils who ask the teacher to repeat the same thing an endless number of times."

Instruction was duplicated in the two classes—the same material presented in the same way by the same teacher—so that neither the teaching nor the learning situation would be a factor in the final analysis.

These classes studied more material than would have been possible in a regular ninth-grade algebra class. In addition to the completion of the text, *A First Course in Algebra*, second edition, by Walter Hart (D. C. Heath and Company, Boston, 1951), added depth was given by studying the following materials:

> Two units of work published by the Commission of Mathematics of the College Entrance Examination Board, 1958:
>
> > "Concepts of Equation and Inequality"; "Informal Deduction in Algebra: Properties of Odd and Even Numbers."
>
> Theory of sets, Venn diagrams, statistics, etc., from: *Modern Mathematics* by Daymond J. Aiken and Charles A. Beseman (McGraw-Hill Company, Inc., New York, 1959).
>
> Simple operations of the slide rule, such as multiplication, division, and square root, using the "Slide Rule Instruction Book" by John Poland, Engineering Instruments, Inc.
>
> Computations in factoring, fractions, graphing, etc., usually covered in advanced algebra. This material was from *A Second Course in Algebra*, second edition, by Walter Hart (D. C. Heath, Boston, 1951).
>
> Making projects for the South Florida Regional Science Fair. (Several of the pupils received recognition for their mathematics projects at the fair.)

To be as objective as possible in this experiment, it was decided that the supervisor of mathematics of the Dade County public schools should administer all of the

standardized tests involved. The Cooperative Algebra Test was administered during the second week of the school year, and a different form of the Cooperative Test at the end of the school year.

In the analysis of the achievement tests of these two algebra classes, the analysis of covariance technique was used by the assistant director of the Research and Information Department of the Dade County public schools. His report states,

> This method automatically adjusts the final test means for any difference between the two groups being compared, as measured by a pretest, at the same time that it determines whether or not there is a statistically significant difference between the final adjusted test means. The unadjusted means for both groups are shown below:

		Cooperative Algebra test	
	I.Q. (Otis)	*Pretest Raw Scores*	*Final Test Raw Scores*
Eighth-Grade Group	122.82	7.97	37.91
Ninth-Grade Group	121.26	8.56	34.44

> Examination of these scores revealed that the eighth-grade group had a little more ability as shown by the average I.Q. of each group but that the ninth-grade group had a greater knowledge of algebra at the beginning of the year. Adjustment in the final test means on the basis of intelligence would lower the eighth-grade final average slightly and raise the ninth-grade final average; using the achievement pretest as the predictor would have the opposite effect. Using both predictor scores, two analyses were made.
>
> In each analysis, the pretest proved to be an excellent predictor. Also, in each analysis there was very little difference between the slopes of the regression lines. Both of these conditions were essential to the analysis of the *amount* of difference between the final test means. The adjustment in these means was slight in both cases.
>
> *There was a significant difference* between the final test means of algebra achievement, as measured by the Cooperative Algebra Test, adjusted for initial differences in achievement, in favor of the eighth-grade group.
>
> There was no significant difference between the final test means of algebra achievement, as measured by the Cooperative Algebra Test, adjusted for initial differences in ability, but what difference there was favored the eighth-grade group.
>
> The adjusted final test means for both groups are shown below:

	Adjusted for initial test achievement	Adjusted for difference in ability
Eighth-Grade Group	38.23	37.51
Ninth-Grade Group	34.13	34.84
	4.10*	2.67†

* Significant at the 1% level of significance.
† Not statistically significant.

To ascertain how much of the elementary algebra the pupils of these two experimental classes had retained over a period of time, the supervisor of mathematics administered the Cooperative Algebra Test in the fall of the 1960–61 school year as these students entered the ninth and the tenth grades.

The analysis of covariance technique was applied to the new data provided on the follow-up test. The report reads,

> Because one pupil in the eighth-grade group and two pupils in the ninth-grade group moved away during the summer, they could not be included in the follow-up testing or analysis. Consequently the pretest means used in this analysis differ slightly from the pretest means used in the original analysis.
>
> Examination of these scores revealed that the ninth-grade group had a slight advantage in achievement at the beginning of the experiment. The effect this had on the final test averages in the analysis was to raise the eighth-grade mean by a very small amount and lower the ninth-grade mean a little.
>
> The analysis revealed that in the September follow-up *there was no significant difference* between the test means of algebra achievement, as measured by the Cooperative Algebra Test, adjusted for initial differences in their knowledge of algebra.
>
> The means for both groups are shown below:

	Cooperative Algebra Test		
	Pretest	*Final Test (Unadjusted)*	*Final Test (Adjusted)*
Eighth-Grade Group	8.09	28.85	29.04
Ninth-Grade Group	8.44	28.94	28.73
Difference			0.73*

* Not statistically significant.

Mean percentiles	Cooperative algebra test		
	9/16/59	5/25/59	9/14/60
Eighth-Grade Group	23.2	97.4	91.5
Ninth-Grade Group	25.6	94.8	91.5

Conclusion

Test results indicate that age was not detrimental to the achievement in elementary algebra. What is probably just as important is the extra incentive and zest for studying these eighth-graders developed. Parents and pupils alike felt that the challenge of the algebra class helped stimulate their thinking in other areas of study.

These students have been scheduled into geometry during the ninth year of their school experience. Their mathematics program has been accelerated one full year. This makes it possible for these students to complete a mathematics course in their senior year which is comparable to most beginning college-level mathematics courses.

Parents, teachers, and school administrators now have evidence upon which to base their decisions regarding the placement of algebra as an eighth-grade curriculum offering for high-ability mathematics students.

Letters to the editor

Dear Editor:

The letter to the editor, page 39 of The Mathematics Teacher for January, 1961, furnishes the basis for some research work for high school students.

Students who have made these interesting constructions will very likely ask the following questions and attempt to find their answers.

(1) Having chosen a unit of measure and a fixed point (such as the origin) on a fixed line (such as the X axis), what irrational numbers can be put into one-to-one correspondence with points of the fixed line by means of geometric constructions made with ruler and compass only?

(2) If a point P on the X axis is not at a rational distance from the origin when $\overline{OP}$ is constructed using a certain unit of measure, will it remain at an irrational number of units from O if the unit of measure is changed? Obviously, there are an infinite number of ways to change the unit of measure so that the number of units in $\overline{OP}$ will be rational. This is a very simple, easily answered, but provocative question.

(3) Can a problem be simplified by changing the unit of measure? The student soon finds that if a change in the unit of measure makes an irrational length rational, it will make a rational length irrational. If the problem involves *only* irrational numbers of the type $n^{\frac{rx+1}{x}}$ where n and x are constant throughout the problem, the problem can obviously be simplified by changing the unit of measure. Other examples can be found.

(4) Are there irrational numbers which cannot be put into one-to-one correspondence with points of the X axis by means of geometric constructions made with ruler and compass? If so, why not?

(5) What kind of numbers are π and e? How can π and e be put into approximate correspondence with points of the X axis, and can this be done accurately? That is, does any point exist on the X axis to which π will correspond in a one-to-one correspondence of the real number system to points of the X axis? Of course many ingenious ways of doing this approximately will come to the minds of the students, but this will lead to the fact that distances on a circular arc and distances on a straight line cannot be measured in the same units.

(6) Are there other numbers which cannot be put into one-to-one correspondence by any method whatever with the points of a straight line?

(7) Having chosen the unit of measure, is there any point P on the X axis such that P cannot be put into one-to-one correspondence with any symbol in the real number system nor approached as a limit by any infinite series of such symbols? Is there such a P that you would have to invent a new number to describe $\overline{OP}$? If there is one such number, there are an infinite number of them.

Perhaps some of these questions are beyond the present abilities of high school students; perhaps not. But they themselves in attempting to answer these questions will discover other questions which they will be able to answer.

To get started on the first question, students might prove that the following "construction" can be made (by means of the geometric theorem quoted on page 39 in "Letter to the Editor" in the January, 1961, issue of The Mathematics Teacher):

Let $\prod_{i=1}^{s} n_i^{\pm j}$ be the product of any s whole numbers, n_i, each raised to a power, $\pm\frac{1}{2}j$ where j may be any whole number, positive or negative. $i=1, 2, 3, \cdots s$. $j=\pm 1, \pm 2, \pm 3, \cdots \pm m$ and Π means "product of."

(continued on page 572)

• HISTORICALLY SPEAKING,—

Edited by Howard Eves, University of Maine, Orono, Maine

Some uses of graphing before Descartes

by Thomas M. Smith, University of Oklahoma, Norman, Oklahoma

During the first half of the seventeenth century at least four students of mathematics and natural philosophy were making use of a simple graphing technique.

In 1618, Isaac Beeckman, while writing in his journal on the subject of "a stone falling in a vacuum," employed this graphing technique to represent uniform acceleration analyzed in terms of what might be called "geometric infinitesimals." His approach suggests certain features of the infinitesimal calculus that Von Leibniz and Newton were to develop later in that century.

In 1637, René Descartes presented an application of the same graphing technique (without Beeckman's infinitesimal "individua," as he called them) in his newly published *La géométrie.* Analytical geometry, as it came to be called, properly owes its essential character to the simultaneous, independent discoveries of Descartes and Pierre Fermat, but like many another advance in scientific thought, its origins antedate the men who made it important.

In the following year, 1638, Galileo Galilei made use of the same graphing technique that Descartes, Fermat, and Beeckman had employed. He used it to provide a mathematical description of uniform acceleration in the physical example of a body in free fall.

All of these men were engaged, when using the same graphing technique, in describing two variables that are simple functions of each other. At that time the technique was neither new nor fully developed. Indeed, it appears to have been then not quite three hundred years old, according to present historical evidence, and it grew out of an even older tradition of purely rhetorical discussion that omitted geometry when discussing certain simple functional variables.

A common pair of functional variables that were being discussed by 1350 in many universities of Europe were "extension" and "intension." Richard Swineshead of Oxford—a logician known among later scholars simply as "The Calculator"—pointed out explicitly, for example, that the extension of a thing could be altered dimensionally without altering the *intensity* of the thing. One could, in the mind's eye, extend hotness or whiteness, for example, without altering the intensity of either the over-all hotness or the hotness at any particular point, if one wished. Or one could increase the intensity without increasing the extent.

Swineshead used a geometric analogy to clarify his point: geometrically speaking, one could place one rectangle beside another without increasing the over-all length, if one chose. Or he could place

one rectangle next to another in such a way as to add to the over-all length but without affecting the width.

Some time between 1345 and 1365, these notions were systematically explored, developed, and sharpened, especially under the mind and the hand of one man, a Parisian scholar named Nicole Oresme. The essence of the technique that he systematized, however, was first employed by an Italian logician, Giovanni di Casali, while Oresme was still a student. Casali remarked in passing, when discussing the traditional topic of "the velocity of motion of alteration," that if one took the example of the quality hotness, one could conceive of a uniform hotness throughout "just as a rectangular parallelogram is formed between two equidistant lines, such that any part you wish is equally wide with another . . . . "

Again, Casali said, " . . . let there be throughout a uniformly difform hotness, such that it is a triangle . . . . "

We are quite unable to say whether or not Oresme read Casali's treatise. But about 1350 or 1360 Oresme wrote a lengthy work on "the configurations of qualities," and in this work he gave a detailed, systematic, and exhaustive explanation of how to use geometric figures to depict the extension and the intensity of any quality. "Although indivisible points or lines do not actually exist," he said in his introductory remarks, "yet it is necessary to picture them mathematically for the measure of things and for comprehending their proportions. Therefore, every intensity successively acquirable is to be imagined by a straight line perpendicularly erected upon some point of space or of the subject of that intensible thing."

Oresme also pointed out that his figures could be used to depict local motion. The perpendiculars represented speed, the base on which they were erected, time, and the area of the enclosed figure, distance. Uniform acceleration from rest would be portrayed by a right triangle, uniform speed by a rectangle.

These same geometric representations of motion were employed by Galileo and Beeckman nearly three hundred years later.

And in the meantime, what had happened to this simple graphing technique? Preliminary evidence indicates it persisted, often in the form of marginal illustrations, in a large number of documents. Thus, at the present time, thirteen extant manuscripts are known of Oresme's long work, *De configurationibus qualitatum*, and two copies are known of another treatise in which he described his graphing technique—his "Questions on the books of Euclid's Elements."

These treatises apparently were never printed after movable type and the printing press became available during the fifteenth century. However, another work, a primer, quite brief, was printed more than once. This little handbook was called "On the latitudes of forms." For a while, modern authorities, such as Pierre Duhem, H. Wieleitner, M. Curtze, Lynn Thorndike, Anneliese Meier, and Marshall Clagett, thought that Oresme had composed the primer. More recently, Miss Meier has suggested it was written by one Jacob of Florence (or Naples or St. Martin). Whoever wrote it, there is no question that it derives from Nicole Oresme's full-length treatise, "On the configurations of qualities."

Twenty-two versions of the short work that we know of are extant. Eighteen of these are manuscript copies; four are printed editions. The earliest dated manuscript is inscribed "1395." The last of the editions was printed in 1515.

Five of the manuscripts were written during the fourteenth century. Eight more copies survive from the fifteenth century; two of these are printed versions, the first dated 1482, the second 1486. Two more printed versions are known to be from the sixteenth century, one published in 1505, the other in 1515.

Seven manuscripts remain that cannot be sharply dated with any assurance at

present. Tentatively, their provenance would seem to place them in the fourteenth century or the fifteenth century.

The primer was the most popular of all the treatises using or discussing the Casali-Oresme figures. However, scattered instances of the use of these simple graphs are to be found in other documents of the fourteenth, fifteenth, sixteenth, and seventeenth centuries. It is not always possible to say with assurance in every case who wrote a particular treatise that contains or refers to the figures embodying the crude graphing technique here under discussion, nor is it possible to assert that the figures in the margin were always placed there by the man who inscribed the text. Nevertheless, the fact remains that at present some fifty documents, written or printed, are known which reveal that this protographing technique was in use among European scholars of the fourteenth, fifteenth, sixteenth, and seventeenth centuries as an aid to their exploration and their understanding of certain abstract concepts of the uniform and the nonuniform.

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The co-ordinate system of Descartes

by Cecil B. Read, University of Wichita, Wichita, Kansas

It is becoming an increasingly common practice to introduce, at an early stage in mathematics education, a co-ordinate system and some elementary work in graphing.

Almost exclusively such an early introduction involves a rectangular co-ordinate system, called by most teachers and many texts a Cartesian co-ordinate system.

The term is technically correct, but unfortunately many students (and some teachers) have the idea that such a system is *the* Cartesian co-ordinate system. Moreover, there seems to be an impression that René Descartes started his approach to analytic geometry in somewhat the manner a current text might do, namely setting up co-ordinate axes, defining concepts such as origin, abscissa, ordinate, etc., and then proceeding to discuss loci. This, while perhaps true under a very broad interpretation, is certainly not true if taken in a literal sense.

Fortunately there is readily available a facsimile of the first (1637) edition of *The Geometry of René Descartes.* Opposite each facsimile page is an English translation.*

Even the casual reader of this book cannot help but be impressed by the extent to which Descartes expresses geometric problems in algebraic terms, and equates the solution of a geometric problem to an algebraic equation and vice versa. This has been called the essence of analytic geometry.

However, the reader looking for a description of a "Cartesian co-ordinate system" or a definition of such terms as origin, abscissa, ordinate, and co-ordinate, will be disappointed. In fact, the reader will perhaps be surprised to find descriptions of methods of drawing curves by use

* David Eugene Smith and Marcia L. Latham, *The Geometry of René Descartes* (Chicago: The Open Court Publishing Co., 1925).

of instruments other than the straightedge and compass.

Perhaps the statement found on page 319 of the original edition (pages 48–49 of the facsimile reproduction and translation) expresses the fundamental concept of analytic geometry:

> . . . all points of those curves which we may call *geometric*, that is, those which admit of precise and exact measurement, must bear a definite relation to all points of a straight line, and that this relation must be expressed by means of a single equation.

Yet this might well be felt by the beginning student in analytic geometry to be rather remotely connected with the rectangular co-ordinate system to which he has been introduced.

It is true that the index mentions the terms ordinate, abscissa, axes, and transformation of co-ordinates. The reference to "ordinate" on page 67 is to a footnote, certainly the French text does not introduce the term in the modern usage. On page 88, the usage of "ordinates" seems to refer to the distance from an axis of a curve to a point of the curve. Likewise, the concept of "abscissa" (mentioned only in a translator's footnote) seems to refer to the distance from the vertex of a conic section and the foot of an "ordinate" (as previously used). To put it another way, the terms ordinate and abscissa seem, if used at all, to be here applied only in the situation where in modern notation an axis of the curve corresponds to the x-axis, and the vertex is at the origin.

Again, on page 95, a translator's footnote implies that a certain line is equivalent to one of the co-ordinate axes, but the original text does not utilize the terminology. Perhaps a closer approach is found on page 51. The statement:

> . . . I choose a straight line, as AB, to which to refer all its points, and in AB I choose a point A at which to begin the investigation . . .

implies an axis and an origin. Likewise, the statement on the same page:

> . . . yet no matter what line I should take instead of AB the curve would always prove to be of the same class . . .

implies (as is noted by the translators) that the nature of a curve is not affected by a transformation of co-ordinates, although the term "transformation" does not appear in Descartes' work. The treatment on page 52 introduces indeterminate quantities y and x which correspond in a sense to our modern ordinate and abscissa (in a rectangular sytem).

By no means do we detract from the contribution of Descartes by pointing out that his treatment and notation are not the same as, nor always directly equivalent to, a modern rectangular co-ordinate system. For example, on pages 26–27, Descartes considers (in the plane) four or even more straight lines given in position (but not with specified length). A point C is referred to these given lines. These lines of reference are essentially co-ordinate axes (note there may be more than two). Descartes then points out that so many lines are confusing. On pages 27–29 we find a figure and an accompanying statement:

> . . . I may simplify matters by considering one of the given lines and one of those to be drawn (as for example, AB and BC) as the principal lines, to which I shall try to refer all the others. Call the segment of the line AB between A and B, x, and call BC, y.

Yet in the statement and the figure, there is no requirement that AB shall be perpendicular to BC; in fact, in the figure this is clearly not the case. In other words, if we (justifiably) imply from this statement a set of two co-ordinate axes, we must not imply a *rectangular* co-ordinate system.

A few modern texts mention the possibility of an *oblique* co-ordinate system, that is, one in which the axes do not meet at right angles. Some students may be interested in the development of formulas which hold when the axes meet at any angle ω, not necessarily 90°. To state the problem somewhat differently: which formulas of elementary analytic geometry, valid for a rectangular co-ordinate system, would also be valid when the axes intersect at an arbitrary angle? One might suggest

as a start a study of the formula for the length of a line segment, and the midpoint formula. For those seeking a problem of considerably greater difficulty, consider the oblique co-ordinate system in three-dimensional space.

Certainly an analysis of Descartes' treatment may correct the false impression sometimes held, namely that in 1637 Descartes presented analytic geometry in its modern form—or the equally erroneous impression that analytic geometry as currently presented is identical with that originally developed by Descartes.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least two months prior to the month in which the issue appears to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

NCTM Convention Dates

FORTIETH ANNUAL MEETING

April 16–18, 1962
Jack Tar Hotel, San Francisco, California
Kenneth C. Skeen, 3355 Cowell Road, Concord, California

JOINT MEETING WITH NEA

July 4, 1962
Denver, Colorado
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D.C.

TWENTY-SECOND SUMMER MEETING

August 23–25, 1962
University of Wisconsin, Madison, Wisconsin
H. Van Engen, School of Education, University of Wisconsin, Madison 6, Wisconsin

Other Professional Dates

Texas Council of Teachers of Mathematics

November 2–4, 1961
University of Texas, Austin, Texas
Dr. Robert E. Greenwood, Mathematics Department, University of Texas, Austin, Texas

Northeastern Ohio Teachers Association

November 3, 1961
Cleveland Engineering and Scientific Center, 3100 Chester Avenue, Cleveland, Ohio
Walter F. Rosenthal, 481 Northfield Road, Bedford, Ohio

Virginia Education Association, Mathematics Section

November 3, 1961
Womens' Club Auditorium, Richmond, Virginia
Simeon P. Taylor, Yorktown High School, 5201 N. 28th Street, Arlington 7, Virginia

Georgia Mathematics Council

November 10–11, 1961
Rock Eagle State Park
Eatonton, Georgia
Martha Rogers, 2908 Macon Road, Columbus, Georgia

New York Society for the Experimental Study of Education; Section 10—Mathematics

December 1, 1961
Washington Irving High School, New York, New York
Mary G. Rule, 58 Spring Avenue, Bergenfield, New Jersey

Women's Mathematics Club of Chicago and Vicinity

December 2, 1961
Henrici's Restaurant, 71 W. Randolph, Chicago, Illinois
Dr. Ruth Ballard, University of Illinois, Navy Pier, Chicago 11, Illinois

The Mathematics Club of Greater Cincinnati

December 2, 1961
Roy D. Matthews, Board of Education, 608 E. McMillan Street, Cincinnati 6, Ohio

A column of unofficial comment

An open letter on teaching machines and programed instruction

by Leander W. Smith, School Mathematics Study Group, Stanford, California

Each generation feels the pressure of time—to do, to make, to remake, and to perfect the world within a few short years. And as each generation leaves its mark, large or small, the history of mankind records and time slips away. The challenge persists. But few generations have been challenged as seriously as that of the present, for time itself seems accelerated.

Science and technology, outgrowths of forces joined for protection and the extension of leisure, now accelerate not only alpha and gamma particles but the very pulse of life itself. We race to be first in space, to make the best better, to outperform and outdo the records of the past, and to educate all children to endure the quickening pace.

To expedite the job of education, technological forces have joined the psychologists and educators in an invasion of the classroom. In they march with films, television, tapes, and now teaching machines. We can only begin to speculate on their impact—a trend from mass education to individual instruction coupled with more students, fewer teachers, and the persistent pressure of time. In a few short years we must learn what the educative process is and how it can be performed. The task demands more of us than our predecessors mastered in centuries of evolution.

The understanding of teaching machines (more properly termed "auto-instructional learning aids") and programed instruction is complex, requiring research and evaluation. Here we shall but scratch the surface and hopefully accept the challenge to dig deeply and learn quickly. The sources of further information cited at the conclusion of this paper will open the door —each must go through himself.

Although devices for self-instruction were invented in the last half of the nineteenth century, it was Pressey (1926) at Ohio State who foreshadowed the teaching-machine movement—but too soon [10].* With the burden of war in the forties came a need for training aids to standardize instruction, to relieve men to tend the business of war. So there developed simple machines to train personnel, for use in war and industry—machines to teach codes, mechanical skills, and rote learning. From this and the surge of electronics came men who saw an opportunity—a light under the educational bushel basket. Meanwhile, some psychologists (Skinner, Holland, and Hilgard, among others) began to re-evaluate the theories of learning and face the realities of an overcrowded classroom with too much to cover and too little time. B. F. Skinner of Harvard began the evolution of training devices with presentation of learning sequences performed with a minimum of error. The initial devices were indeed developed to reduce the amount of drill and routine instruction

* Numbers in brackets refer to the references listed at the end of the article.

(testing, etc.) and to relieve the demands on teacher time.

As time waned and necessity grew, there arose a cry for instructional devices which, like traditional texts of the past, presented uniform content but could be used by students at varying rates of speed without an instructor present. Like a nuclear reaction, the teaching-machine movement sprung from simple paper-fed devices to electronic multi-component computer-audio-visual units.

And as one pendulum swung into the generation of hardware, another moved to the program itself, that is, the material contents of the machines. Punchboards, tab multiple-choice devices, disc and roll machines, chemically treated score sheets (pen turns a spot red if the answer is wrong, green if it is right), complex computer centers, audio-visual units, and a variety of combinations on the one hand are challenged by programers who maintain linear and branching courses of study with little or no use of machines.

Although it may hurt our pride, we can no longer pose the academic question "Can machines (programs) teach?" Film producers, tape recorders, record makers, television directors, and now teaching-machine programers realize the fact that there is a service they can render. Teachers cannot assume roles like Aristotle or Newton, knowledge has grown too complex for that. We recognize, too, that all teachers cannot be exposed to the great minds—the master teachers. So, like it or not, we resort to audio-visual techniques to extend the exposure of our students. But teaching machines? Let's examine their key points and a few implications.

Historically, certain aspects of learning, not all known or yet fully exploited, can best be transmitted by specification of the desired behavioral outcomes and presentation as small bits of sequentially ordered information. Those subjects which lend themselves to this presentation are currently the areas explored by programers. Mathematics, some science, grammar, spelling, and foreign languages are most frequent in the research projects. Concept formation and creativity still provide a challenge to this process. Point one, then, concerns the *specification of behavioral outcomes and the sequential ordering of "bits" of information*. To this end, a traditional, well-written text may fill the bill.

To preclude passive reading or scanning of material, the program necessitates activity. "Turn the page," "Push the button," "Write your answer here," and other directions evoke a response from the student. He is busy—with electronic models he may be timed—he is active. And according to some psychologists, his activity augments his learning. Thus we see that point two concerns *active participation by the learner*.

Once his response is made, then what? Does the machine or program say "That's right" or "Sorry, try again"? Some models do, believe it or not. Some devices reward a series of correct responses with candy pellets. The simple machine or program is merely advanced to allow the learner to see the correct answer. Perhaps this is on the next page [6, and 3], or six pages back [4], or under a plastic tab [14]—but no matter where it is, the reinforcement of a correct response is made. Some machines allow a student to advance to the next question if the response is correct, lock until a correct response is selected (see Atronic Tutor, or Astra Auto-Score), or branch to a corrective section (see Western Design Auto Tutor), then return to the question on which an error was made. Point three, then, is that there should be *reinforcement* (preferably immediate) *of correct responses*.

Issues which influence programers in writing their materials that should be carefully considered by potential users of programed instruction include: cueing, error rate, vocabulary control, fading (gradual withdrawal of stimulus), and validation. Programing requires a detailed analysis of subject content and sequence. It is no easy task to write a pro-

gram, and no easier to evaluate one.

There is research going on and it must continue, then be validated and scrutinized by potential users. Time would rush us to make a decision; but we hold in our hands more valuable things than machines—children whose education depends on careful evaluation and preparation.

While the impact of this new media is yet to be felt, even at this infant stage programed instruction and teaching machines have implications of no little concern. Possibilities, and those enumerated here are by no means exhaustive, include:

1 the establishment of "machine rooms" in schools where students can go for self-instruction in a remedial reinforcement or acceleration program;
2 programed texts [3, 4, 6] for use without machines;
3 better standard texts, pretested in programed form or written with programed sections on those topics found most suitable to this medium of communication;
4 combinations of audio-visual programed approaches used by teachers.

Whatever is ultimately evolved or derived, there are sources to examine *now*—research to do—testing to be done—and in the end, objective evaluation of where we are, have been, and may be going.

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15 *TMI-Grolier.* Teaching Machines Incorporated, Albuquerque, New Mexico. Programs in elementary algebra, statistics, Hebrew, music, and spelling. Unit cost is about $7.50.

Letters to the editor

(*continued from page* 564)

Next, the student might try constructing his own apparatus for the "construction" of p^3, where p is any prime number.

These experiments lead on into questions two through seven and then on without limit—we hope into more number theory.

The following references should be of help to the student in his research:

Richard Courant, *What Is Mathematics?* Part 1, Fundamental Geometrical Constructions; Construction of Fields and Square Roots; Constructible Numbers and Number Fields.

Tobias Dantzig, *Number, the Language of Science*, Chapters 6, 7, 8, 9, 10, 11, 12.

J. A. BARTMAN
Albuquerque, New Mexico

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

BOOKS

The Child's Conception of Geometry, Jean Piaget with Barbel Inhelder and Alina Szeminska, translated from the French by E. A. Lunzer (New York: Basic Books, Inc., 1960). Cloth, 411 pp., $7.50.

In these exciting days when the high schools are in transition to a point-set interpretation of secondary-school geometry and when revision of elementary-school mathematics is including an extensive development of intuitive geometry, reading *The Child's Conception of Geometry* is a stimulating experience, sometimes enlightening, frequently exasperating, but never trivial.

The Child's Conception of Geometry is a sequel to *The Child's Conception of Space* (Piaget with Inhelder, 1956), and in these, as in twelve other volumes produced since 1926, Piaget has set himself the gigantic task of revealing origins and developments of great concepts basic to man's intellectual and cultural growth. This research is not armchair thinking extrapolated from a few formally designed, highly specialized experiments. Piaget and his staff go directly to children for evidence. Tremendously clever problem situations have been designed—situations in which the child can reveal evidence of possessing the concept under study. Children are tested one at a time. Each child's answers are recorded and his behavior described. Occasionally one could wish for supplementary diagrams or photographs, but usually the dialog and descriptions are sufficient for vivid interpretation. The children studied range in age from 1 year, 7 months to 13 years, 4 months.

Each series of interview records is introduced by the authors' announcement of what is about to be illustrated and followed by a detailed interpretation of the events recorded. It is in these sections that the geometry-teacher-reader's exasperation reaches explosive dimensions. The authors frequently use mathematical terms,—e.g., 'co-ordinate system', 'the laws of mathematical groups', 'nested intervals', 'transitivity', 'measure', 'metric space', 'equality'—in a plausible way, and the reader is tricked into thinking that he is experiencing communication, only to be brought up short by a line on the next page that will not "fit." Sometimes the discourse is a cloud of impenetrable polysyllabic dust. But frequently, by backtracking to earlier pages and doing some energetic cryptanalysis, the reader can discover that something exciting is being revealed.

As one begins the book, such terms as 'co-ordinate system', 'metric space', 'equality', 'congruence', and 'mathematical group' seem to promise the reader that he is to see the child's conception of geometry traced all the way through to awareness of the number plane, a concept prerequisite to treatments of Euclidean geometry such as Veblen's or Vaughan's (and a concept shown to be available to twelve-year-olds in at least one of the school mathematics curriculum revision projects in the United States). But all too soon the reader finds that 'co-ordinate system' refers merely to two intersecting geographical paths which provide a "reference system" for locating neighboring landmarks on a drawing; that having the concept called 'metric space' is awareness of a procedure by which the length of one object can be described in terms of the length of another object; that equality for intervals is the same thing as congruence; that congruence has nothing to do with a distance-preserving matching between sets of points; and that 'the mathematical idea of a group' refers to something exemplified by "grouping of changes of position" according to before (or after) in the memory of one's trip from home to school. The list could go on and on. In the chapter entitled 'Angular Measurement', page 82, we find, " . . . the measurement of an angle is no less a two-dimensional operation than the definition of a point in a rectangular plane area." The suspicions aroused on page 82 are confirmed in the conclusions on page 406: " . . . they [children] can measure angles which are also plane surfaces, since an angle may be regarded as a plane sector delimited by its arms. But areas are measured only in terms of units which are themselves areas and angles only by the superposition of similar angles." So we go back and reread, knowing that geometry for these authors in this book is restricted to a study of physical space.

It may well be admitted that if there had to be a choice, it would be better for those engaged in revision of the school mathematics curriculum to comprehend a child's development of the physical space concepts prerequisite to abstraction without being aware of the mathematical goals—better this than for the curriculum revisers to possess the concepts of an abstract geometry without recognizing experiences from which the concepts grow. But for trustworthy results in psychological research the investigator should have both. A psychologist investigating "the child's concept of geometry" should be aware of the mathematical entities

named by the mathematical terms he employs. Otherwise, communication is blocked until the mathematician, whose help the psychologist needs, discovers that technical vocabulary of modern mathematics is being subjected to far-fetched metaphorical uses.

The question sequences employed in Piaget's interviews are extremely dynamic in their effect on children. The questions *teach*. And *what* they teach is almost completely bounded by what was in the mind of the psychologist who composed the questions. Furthermore, the conclusions reached by the authors concerning a child's development of the concept under study are also bounded in the same way. Just as some of us have suspected that we would find great changes in both the interviews and the conclusions of *The Child's Conception of Number* (Piaget, 1952) if the author had been aware of the Frege-Russell concept of natural number, so do we now suspect that *The Child's Conception of Geometry* would be different if Piaget, Inhelder, and Szeminska were versed in the point-set interpretation of Euclidean geometry. Early in the book we find that a maplike representation of remembered "changes of position" on the way to school is their test for what they consider the most primitive stage in geometric concept development. But any stranger who has tried to find his way about in the vicinity of Boston, Massachusetts, for example, knows that the natives of such a region get along very well without any such mapmaking ability; and among these same people are many with a clear and well-advanced conception of co-ordinate geometry. The concept is *not* bound to personal travel on the surface of the earth.

The limitations cited in this review do not belittle the work of Piaget and his associates. These scholars are showing us how to conduct genuine investigations into great problems which had remained elusive until they were subjected to Piaget's inspired approach. No matter how much of Piaget's monumental work may need revision, extension, or correction, his approach and the kind of evidence on which he depends are a refreshing, invigorating, enlightening substitute for academic discussions of child development. *The Child's Conception of Geometry* is reaching us in translation at an opportune time for mathematics education in the United States.—*Gertrude Hendrix, University of Illinois, Urbana, Illinois.*

The Teaching of Secondary Mathematics, 3rd ed., Charles H. Butler and F. Lynwood Wren (New York: McGraw-Hill Book Company, Inc., 1960). Cloth, ix+624 pp., $7.50.

This is the third edition of a book that is already well known by teachers of secondary mathematics and by university personnel in charge of the training of secondary-mathematics teachers. It is this reviewer's conviction that the authors in preparing this edition have progressed from a very good second edition to an outstandingly good third edition. After reviewing this conservative, ultra up-to-date, scholarly presentation, one strongly feels that no secondary or even college teacher of mathematics, who wants to get a clear picture of secondary mathematics as it has been and as it probably will be in the near future, can afford to be without this excellent book.

In Part I, entitled "The Program and the Improvement of Instruction in Secondary Mathematics," the authors state, "We have tried to bring together some of the most significant contemporary thought respecting the program in secondary mathematics and the broad problems involved in making it effective." After reading the 279 pages of completely rewritten material in which the authors deal with the evolving secondary-mathematics program, the impact of modern mathematics, methods of stimulating and maintaining interest, effective teaching and guided learning, supervision and evaluation of instruction, and the professional preparation of teachers, one feels that the above objective has been fulfilled in a most effective manner.

Part II of the book deals both with subject matter and with the teaching of the specialized subject matter of secondary mathematics. In this section, the authors had two main objectives in mind, "One has been to retain the material of the second edition that we believe will continue to be helpful to teachers; the other has been to exhibit and illustrate, in the context of familiar subject matter, some of the newer concepts and newer emphases in teaching that we believe will soon be reflected in mathematics textbooks and courses of study for secondary schools."

The inevitable first step in any improvement process is confusion. Certainly secondary mathematics at the present time is in a high state of confusion. In the midst of this confusion it is most interesting to observe how the authors have held, in a very consistent and sensible manner, to a firm middle-of-the-road organization and development of materials. They have done this by magnifying the important values of our present mathematics and at the same time showing how our present arithmetic, algebra, geometry, trigonometry, analytic geometry, and calculus can be improved by the judicious use of the latest modern mathematical concepts.

The approximately 600 thought-provoking questions in the exercises at the end of the twelve chapters add greatly to the value of this book. These questions also make the book an excellent text for use in training teachers of secondary mathematics.

Another outstanding feature is the 74 pages of bibliography appropriately placed throughout the book. This is one of the best bibliographies of secondary-school mathematics available anywhere.

This reviewer would like to be critical to some extent. On page 297, line 6, and page 391, line 38, the authors use the expression "reducing

fractions." Actually, a given fraction can never be reduced; only its terms may be changed. Also on page 313, line 7, "arithmetic mean" has been used when the word "mean" alone would have been better. "Arithmetic mean" is the average of two numbers and should be used only in connection with arithmetic series. On page 397, line 40, and page 411, line 28, a question is raised concerning the word "quantity." Are "quantity" and "number" identical? Finally, on page 462, line 25, why are a pair of compasses called a compass? Does one call a pair of scissors a scissor? A pair of pliers a plier? A pair of tweezers a tweezer?

The format is good, and the book is remarkably free from errors. However, on page 385, in Figure 14-2, the graph looks more like a graph for $y \geqq x$ then it does for $y > x$.

Any teacher of mathematics who desires to be a better teacher can find expert guidance and help in reaching his goal in this very valuable contribution to education.—*William A. Gager, University of Florida, Gainesville, Florida.*

FILMS

Axioms in Algebra, color, 12 minutes, $135. International Film Bureau, Inc., 332 S. Michigan Ave., Chicago 4, Illinois.

Have you read any of the recent literature dealing with modern mathematics or investigated any of the proposals made by various groups working on improving the mathematics curriculum? If so, you might find the title of this film somewhat misleading and the content disappointing. I did.

I am sure much of my disappointment was due to an expectation that this new film would touch on algebra as an axiomatic system. Instead, the "Axioms in Algebra" turn out to be the four familiar rules for solving equations mechanically. The first axiom presented is the addition axiom: "If equal quantities are added to equal quantities, the results are equal." The subtraction, multiplication, and division axioms are similarly worded.

Today we are told to use precise language in dealing with mathematics and to use symbols in a carefully defined manner. But, at one point in the film, we see '$x-5+5=3+5$,' and we hear, "Minus five and plus five equals zero." At another point, our attention is drawn to '$+2-2$' in an equation, and the narrator says, "These two equal zero." In another situation involving the solution of an equation, we are told how the man in the situation has handled the problem. "He has multiplied each side by *an equal number*." The symbol '=' would be difficult to define as used in this film. For example, the following is seen (from left to right): a picture of a loaded coal truck, a '−', a picture of an empty coal truck, and then '$=15{,}000-7{,}000$'. In addition to this somewhat careless use of the symbol for equality, the '=' is also used in an acceptable manner in equations.

This film has many good features, and those who believe my remarks above to be unimportant or trivial may find it a valuable addition to their stock of teaching aids. Effective use is made of color, and as might be expected of a new film, the quality of both color and sound is excellent. Each of the four axioms is introduced in a way that should be appealing to students. The first action in the film is a sequence showing a mouse being shot into space in the nose cone of a rocket and later, after recovery of the nose cone, being tested for the effects of the trip. This testing of the mouse leads into the solution of an equation using the addition axiom. The remaining three axioms are introduced in situations in which equations are used to solve everyday problems. The final sequence presents a challenge to the student to select and apply the appropriate axiom for the solution of a problem which should be of interest to most students because it concerns the selection of phonograph records for a party.

An alert class will undoubtedly notice things about which they will want to ask questions. Such topics as extracting roots and raising to powers are mentioned briefly, just enough to arouse curiosity.

Axioms in Algebra appears to be a film which should be interesting and instructive to students who are just beginning the study of simple linear equations in the traditional first course in algebra. I was disappointed in this new film because the approach is not the approach recommended by the various groups working on curriculum improvement, and because some carelessness in the use of terms and symbols is present.—*Martin S. Wolfe, University High School, Urbana, Illinois.*

"State supervisors in science, mathematics, and foreign languages have increased in number."—*From "Schools in Our Democracy," Office of Education, U.S. Department of Health, Education, and Welfare.*

• TIPS FOR BEGINNERS

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

1024 tosses

by Marvin C. Volpel, State Teachers College, Towson, Maryland

Near the end of the semester, the students in my class in basic statistics are required to conduct an organized investigation which will employ many of the techniques of statistics discussed in the course. The purpose of the project is threefold: (1) to summarize the course in preparation for the final examination, (2) to select and organize the facts of an investigation in project form, and (3) to uncover new truths or generalizations or to verify old ones.

When the binomial distribution was presented in class, we discussed the probability of "heads" appearing when coins were tossed. It was easily shown that when one coin is tossed, the chances of obtaining one head is 1/2. When two coins are tossed it was shown that the probability of obtaining two heads is 1/4, of obtaining only one head is 2/4, and of obtaining no heads, 1/4. If three coins are tossed, the probabilities of obtaining three heads, two heads, one head, no heads are 1/8, 3/8, 3/8, and 1/8, respectively.

The expansion of $(p+q)^n$ was developed and discussed with the following definitions; p = probability of success, q = the probability of not-success, and n = the number of items in the experiment. In the analysis of coin tossing, we discussed the probability of certain happenings when 10 coins are tossed. Since $p=q=1/2$, then $(p+q)^{10}$ represents the sum of the probabilities of the several events. The probability of obtaining 10 heads is one in 1024 tosses of 10 coins, the probability of getting 9 heads is 10 in 1024, and so on. The expected probabilities are tabulated in Table 1.

One of my students had serious doubts about the binomial distribution and set out to show that in real-life situations of coin tossing the actual behavior would not follow the frequencies obtained in the expansion of $(p+q)^n$. He selected 10 un-

TABLE 1

Number of heads	10	9	8	7	6	5	4	3	2	1	0
Probability of the event happening	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$
Expected frequency in 1024 tosses	1	10	45	120	210	252	210	120	45	10	1
Observed frequency in 1024 tosses	0	9	47	111	216	254	216	119	43	8	1

biased coins and over a two-week period tossed them 1024 times. He was amazed and somewhat displeased with—though proud of—his discovery, for contrary to his belief the actual results of his experiment were almost identical with those derived by the binomial expansion. The results of his experiment compared with the expected frequencies and probabilities should be of interest to the reader. This information is shown in tabulated form in Table 1.

Mathematical bingo

by Patricia Ann Harris, Mount St. Scholastica College, Atchison, Kansas

An interesting and challenging game for a mathematics club is an adaptation of the familiar bingo. Prepare a series of thirty cards each 5×6 inches from show card or some other reasonably stiff material. On each card mark six rows and five columns as shown in the accompanying diagram. In the spaces on the top row, mark in large characters B I N G O.

For entries on the cards, work out one hundred mathematical equations ranging from simple to fairly difficult, depending on the mathematical training of the players. Some sample equations that may be used are:

a) $\{[3, 7] \cup [2, 4]\} = A$
(Solution: $[2, 7]$)

b) $((22 \times 4) + 3) \bmod 5 \equiv x \bmod 5$
(Solution: 1)

c) $(3x+1)^2 - 7(3x+1) + 12 = 0$
(Solutions: 1, 2/3)

Separate these equations into five distinct groups and label the groups B, I, N, G, and O. Number the equations in each group from 1 to 20.

The next step is to mark the solutions to the prepared equations in the squares on the cards in the columns corresponding to the group labels of the equations. For example, select entries for column "B" from the group of equations labeled "B." Mark the third space under "N" on each card "Free." Make sure that no two of the equations in any one group have the same answer.

We are now ready to play the game. Each player has been provided with a supply of lima beans to cover the correct solutions and with a scratch pad and a pencil. The caller takes his master sheet on which the hundred equations and answers are listed and proceeds to call off each equation, in turn, under its proper letter. Thus, for the first example, given previously, the caller would direct: "Under N, the set of integers from 3 to 7 union the set of integers from 2 to 4." The player would search his card under column "N" for the set of numbers 2 to 7 inclusive, written $[2, 7]$.

This process is repeated until a player has completely covered a row, a column, or a diagonal, in any of which events he calls out "Bingo!" The player then reads back the letter and the number (see note under the diagram) of each covered square and the caller checks them on the master chart to see that the problems have been worked correctly.

If it is desired, mathematical prizes may be awarded to the winners. To make the game enjoyable, the caller must allow ample time for the players to determine the solution to one problem before calling a second one.

In preparing the bingo cards, use only those equations with solutions that are

fairly simple and compact in form to fit each solution in the square allotted to it. Mathematical bingo is adaptable to any grade level and might prove a stimulating form of review for various units of work during the year.

B	I	N	G	O
3 -1	16 $0, 1, -1$	9 $0, -1/2, -6$	1 1 mod 5	4 $p \to q$
5 21.8	9 $\cot \theta$	20 2, mod 3	16 $3/2, -4$	3 21.7
2 $\frac{2bi}{a^2+b^2}$	1 10.6	Free	4 4, mod 5	10 1
10 -2	4 16	16 $[2, 6]$	9 -2	15 -1
15 $3/2, 4$	20 $1/2, 1$	3 $[3, 4]$	5 $x=y$	2 $5-i$

Sample Card

Note: The numbers in the upper left of each square are merely to help the caller in checking the solutions in case of a "Bingo."

Announcement

The National Commission on Teacher Education and Professional Standards of the National Education Association is seeking to determine the extent of possible interest in a placement-information type of service concerning openings in teacher-education institutions and concerning the availability of personnel.

The proposal is to announce during 1961–62 through the columns of the *Journal of Teacher Education*, in a highly ethical and confidential manner, (1) openings in teacher-education institutions and (2) the availability of personnel for employment in teacher-education institutions. If you wish to learn more about the proposal, please write to *Journal of Teacher Education*, NEA, 1201 16th Street, N.W., Washington 6, D.C., *enclosing a self-addressed, stamped envelope.*

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Registrations at NCTM meetings

Perhaps the most significant development in the attendance at NCTM conventions during the 1960–61 year was the unusually large attendance at the Thirty-ninth Annual Meeting, with an official registration of 3,232. This figure was nearly 1,200 greater than the registration at the largest previous annual meeting.

Below are official registration reports for all of the NCTM meetings held during the 1960–61 year. The total number of persons present at the site of each meeting was somewhat greater than the number reported here for two reasons. (1) The official registration report does not include members of families or friends who did not come primarily to attend the meeting. (2) There are always a few convention participants who fail to take the time to register formally.

Registrations at the Twentieth Summer Meeting

The National Council of Teachers of Mathematics, Salt Lake City, Utah, August 21–24, 1960

Arizona	12	New Hampshire	2
Arkansas	1	New Jersey	10
California	76	New Mexico	2
Colorado	24	New York	14
Connecticut	2	North Carolina	2
District of Columbia	8	Ohio	17
Florida	2	Oklahoma	3
Georgia	3	Oregon	17
Idaho	7	Pennsylvania	5
Illinois	56	Rhode Island	2
Indiana	16	South Carolina	1
Iowa	10	South Dakota	1
Kansas	17	Tennessee	2
Louisiana	7	Texas	10
Maine	2	Utah	178
Maryland	4	Virginia	2
Massachusetts	10	Washington	16
Michigan	14	Wisconsin	13
Minnesota	7	Wyoming	9
Missouri	10	Canada	9
Montana	21	Foreign	7
Nebraska	7		
Nevada	1	TOTAL	639

Registrations at the Nineteenth Christmas Meeting

The National Council of Teachers of Mathematics, Tempe, Arizona, December 28–30, 1960

Arizona	260
Arkansas	1
California	111
Colorado	18
Connecticut	4
District of Columbia	5
Florida	2
Georgia	1
Illinois	36
Indiana	9
Iowa	7
Kansas	3
Louisiana	2
Maryland	2
Massachusetts	4
Michigan	5
Minnesota	6
Missouri	13
Nebraska	3
Nevada	3
New Jersey	3
New Mexico	12
New York	6
North Dakota	4
Ohio	5
Oklahoma	3
Oregon	7
Pennsylvania	6
Rhode Island	1
Tennessee	5
Texas	21
Utah	1
Virginia	1
Washington	1
Wisconsin	5
Canada	4
Foreign	1
TOTAL	581

Registrations at the Thirty-ninth Annual Meeting

The National Council of Teachers of Mathematics, Chicago, Illinois, April 5–8, 1961

Alabama	9
Arizona	3
Arkansas	12
California	46
Colorado	12
Connecticut	26
Delaware	4
District of Columbia	23
Florida	29
Georgia	12
Idaho	2
Illinois	1,319
Indiana	211
Iowa	63
Kansas	32
Kentucky	12
Louisiana	13
Maine	1
Maryland	29
Massachusetts	45
Michigan	197
Minnesota	83
Mississippi	6
Missouri	165
Nebraska	15
New Hampshire	4
New Jersey	50
New Mexico	2
New York	108
North Carolina	7
North Dakota	10
Ohio	183
Oklahoma	22
Oregon	3
Pennsylvania	90
Rhode Island	2
South Carolina	3
South Dakota	4
Tennessee	40
Texas	27
Utah	2
Vermont	1
Virginia	26
Washington	3
West Virginia	7
Wisconsin	228
Wyoming	1
Canada	38
Foreign	2
TOTAL	3,232

Registrations at the Joint Meeting with the National Education Association

The National Council of Teachers of Mathematics, Atlantic City, New Jersey, June 28, 1961

Alabama	3
Alaska	1
Arizona	1
Arkansas	1
California	10
Colorado	1
Connecticut	5
Delaware	10
District of Columbia	2
Florida	2
Georgia	3
Hawaii	2
Idaho	2
Illinois	16
Indiana	5
Iowa	12
Kansas	4
Kentucky	3
Louisiana	2
Maryland	14
Massachusetts	8
Michigan	10
Minnesota	1
Mississippi	6
Missouri	4
Nebraska	2
New Jersey	78
New York	24
North Carolina	3
Ohio	20
Oklahoma	7
Oregon	3
Pennsylvania	54
Rhode Island	1
South Carolina	3
South Dakota	2
Tennessee	4
Texas	5
Utah	2
Vermont	2
Virginia	11
Washington	4
West Virginia	6
Wisconsin	1
Wyoming	1
Puerto Rico	1
TOTAL	361

Letters to the editor

Dear Editor:

There are times when it seems helpful to be able to assign a number of significant digits to a measurement, say, n/d units where n and d are natural numbers and $1/d$ is the precision unit. For example, how many significant digits does the measurement 4 8/16 inches or 72/16 inches have? We digress for a moment to point out that assigning a number of significant digits is a rough way of indicating the accuracy of the measurement. A precise way of indicating accuracy is by means of the relative error of the measurement. Moreover, one would be inclined to say that if two measurements have the same relative error then they should have the same number of significant digits, but not conversely.

Let's come back to our example. The relative error of the measurement 4 8/16 inches is $1/32 \div 72/16 = 1/144$. Recall that the relative error of a measurement is the maximum error divided by the measurement. Note that the relative error of 72 inches, where the precision is 1 inch, is also 1/144. The general situation is readily obtained. The relative error of n/d units where $1/d$ units is the precision, is $(1/2d) \div (n/d) = 1/2n$, the very same as the relative error for n units, where the precision is 1 unit. Thus, the number of significant digits that one would assign to n/d inches where the precision is $1/d$ inches should be the same as the number of significant digits that n inches has, where the precision is 1 inch. Hence, the number of significant units in the measurement 4 8/16 inches $= 72/16$ inches is the same as the number of significant digits in 72 inches, namely 2.

In conclusion, we may say that the number of significant digits in any measurement is the number of digits in the numeral that indicates the number of precision units in the measurement. It is understood that all numbers must be expressed in decimal notation: numerators, denominators, and the whole number component associated with a "mixed" number.

Sincerely,
HARRY D. RUDERMAN
Chairman of Mathematics Department
Hunter College High School
New York, New York


A circular cylinder is one whose bases are circles.
The altitude is the perpendicular distance between the bases of a cylinder.
The axis of a cylinder is a line connecting the centers of the bases.
A right section of a cylinder is the figure formed by a plane perpendicular to an element and cutting all the elements.
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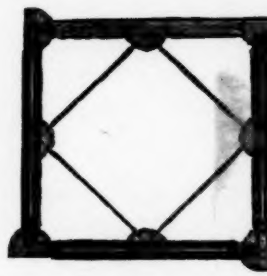

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